

Mirror Symmetry of elliptic curves and Ising Model

Shi-shyr Roan ¹

Institute of Mathematics

Academia Sinica

Taipei , Taiwan

(e-mail: maroan@ccvax.sinica.edu.tw)

Abstract

We study the differential equations governing mirror symmetry of elliptic curves, and obtain a characterization of the ODEs which give rise to the integral \mathbf{q} -expansion of mirror maps. Through theta function representation of the defining equation, we express the mirror correspondence in terms of theta constants. By investigating the elliptic curves in X_9 -family, the identification of the Landau-Ginzburg potential with the spectral curve of Ising model is obtained. Through the Jacobi elliptic function parametrization of Boltzmann weights in the statistical model, an exact Jacobi form-like formula of mirror map is described .

¹Supported in part by the NSC grant of Taiwan.

1 Introduction

Recent progress in physicists' construction [6] [17] of the "number" of rational curves of an arbitrary degree on a large class of Calabi-Yau spaces has stimulated efforts to find a mathematical understanding of this remarkable "counting" principle. As is known the main ingredient of practically all examples is to express the "counting" function, called the mirror map, in terms of solutions of a generalized hypergeometric system. As a physical theory, it is the $N=2$ supersymmetry (SUSY) two-dimensional Landau-Ginzburg (LG) models to describe the mirror symmetry of σ -models on Kähler manifolds with vanishing first Chern class, (for the basic notion of mirror symmetry, we refer readers to [17]). This novel principle gives also counting functions on other $c_1=0$ algebraic manifolds of an arbitrary (complex) dimension. In the elliptic curve case, there are three ways of realizing them as hypersurfaces in weighted projective 2-space, and the moduli parameter is always connected to the classical J -function by an algebraic relation [8] [9]:

	Constraint	Differential operator	$1728J(z)$
P_8	$x_1^3 + x_2^3 + x_3^3 - z^{-1/3}x_1x_2x_3 = 0$ in $\mathbb{P}_{(1,1,1)}^2$	$\Theta^2 - 3(3\Theta + 2)(3\Theta + 1)$	$\frac{(1+216z)^3}{z(1-27z)^3}$
X_9	$x_1^4 + x_2^4 + x_3^2 - z^{-1/4}x_1x_2x_3 = 0$ in $\mathbb{P}_{(1,1,2)}^2$	$\Theta^2 - 4(4\Theta + 3)(4\Theta + 1)$	$\frac{(1+192z)^3}{z(1-64z)^2}$
J_{10}	$x_1^6 + x_2^3 + x_3^2 - z^{-1/6}x_1x_2x_3 = 0$ in $\mathbb{P}_{(1,2,3)}^2$	$\Theta^2 - 12(6\Theta + 5)(6\Theta + 1)$	$\frac{1}{z(1-432z)}$

Table (I)

Here the differential operator describes the Picard-Fuchs equation for the family, and $\Theta := z \frac{\partial}{\partial z}$. With the variable t obtained by a ratio of fundamental solutions of the differential equation near $z = 0$, the mirror map yields the following numerical expansion of q ($:= e^{2\pi i t}$) for the parameter z :

$$\begin{aligned}
 P_8 : z(q) &= q - 15q^2 + 171q^3 - 1679q^4 + 15054q^5 - 126981q^6 + \dots \\
 X_9 : z(q) &= q - 40q^2 + 1324q^3 - 39872q^4 + 1136334q^5 - 31239904q^6 + \dots \\
 J_{10} : z(q) &= q - 312q^2 + 87084q^3 - 23067968q^4 + 5930898126q^5 - 1495818530208q^6 + \dots
 \end{aligned} \tag{1}$$

Note that the "counting" numbers in these expansions are all integers. For a general Calabi-Yau hypersurface family in a weighted projective space, one also produces a "counting" function of such kind. However the mathematical reason for the arithmetical nature of "counting" functions is poorly understood, but a fundamental understanding of the counting principle should be important to further mathematical development of mirror symmetry. In [8], the generalized Schwarzian equations were derived for mirror maps of one-modulus cases as one effort towards this direction. The starting point of the present work is to clarify the role of differential equations in the integral property of the counting function $z(q)$. We find a characterization of the equations appeared in Table (I) by their qualitative relations with the J -function. For the precise statement of the result, see Theorem 1 of the context. On the other hand, the numerical evidence has also suggested $z(q)$ might possess a certain structure as modular functions. To the author's knowledge, not much is known about the exact modular form-like expression of $z(q)$, even on elliptic curve cases. In this paper, we have obtained the elliptic theta function parametrization of the constraint, i.e. LG superpotential, in Table (I), and also the exact formula of $z(q)$ in terms of theta constants. The key ingredient is the observation of the connection between discrete symmetries encoded in the constraint, and their hidden theta function (projective) representations. Our purpose here is to extensively analyse the discrete symmetries appeared in Table (I), and to determine the theta function parametrization

of the superpotential for each case, which allows one to obtain the exact formula of the moduli parameter. One main contribution of the present work is that we have connected X_9 -family with Ising model, a standard physical theory which has been served as a basis to provide a simple 2-dimensional statistical model. Here the Jacobi elliptic parametrization of Boltzmann weights in Ising model is used for the derivation of theta function representation of the X_9 -potential, and the Jacobi form expression of temperature-like parameter of Ising family leads to a closed form of $z(\mathbf{q})$ for X_9 in (1) in terms of theta constants. With this novel phenomena, it becomes increasingly interesting in the interplay of geometry of $c_1=0$ Kähler manifolds and other 2-dimensional solvable statistical models. As it is well known, theta function parametrizations have provided a powerful tool in 2-dimensional lattice models to obtain quantities of physical interest [3] [15]. In recent years, there has been considerable progress in the study of chiral Potts N -state models [1] [5], as a generalization of Ising model. The Boltzmann weights of the chiral Potts models lie on hyperelliptic curves with a large number of discrete symmetries, and their theta function parametrizations are known in [4] [13]. The question that we address for future investigation is to establish a connection between this hyperelliptic function parametrization with mirror map of Calabi-Yau spaces. A resolution might point towards some future structure, yet to be explained.

The following is a summary of the contents of this article: In Sect. 2, we recall some basic facts on elliptic theta functions and Heisenberg group representation, which will be needed for the discussion of this paper. In Sect. 3, we study the Schwarzian equations satisfied by the mirror map, which are derived from a special type of Fuchsian differential equations [8] [10]. We characterize the differential operators in Table (I), which are solely governed by the integral property of the \mathbf{q} -expansion of the Schwarz triangle function, and its qualitative relation with J -function. Also we indicate the J_{10} -family as an equivalent version of Weierstrass form of elliptic curves. In Sect. 4, the elliptic theta function parametrization of P_8 -family is derived, so is the expression of $z(\mathbf{q})$ in terms of theta constants. Based on the identification of symmetries of the defining equation with the finite Heisenberg group of degree 3, the standard theta function representation of the group gives rise to the parametrization of P_8 -potential. In Sect. 5, we give a brief review on elliptic curve theory related to Boltzmann weights of Ising model, which will be relevant to our discussion. Primary focus is on its Jacobi elliptic function parametrization. With this parametrization, by examining the relation between X_9 -potential with Ising model we derive the Jacobi elliptic function representation of elliptic curves in X_9 -family in Sect. 6, and also the exact formula for the moduli parameter $z(\mathbf{q})$. After carrying out the mathematical results of this paper, finally in Sect. 7 we will mention a comparison of some essential structures in two physical theories: $N=2$ SUSY LG theory and exactly solvable statistical model, the geometry of which is respectively presented in Calabi-Yau spaces and hyperelliptic curves of chiral Potts models.

Notations

$\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ the complex upper-half plane, which is acted by $SL_2(\mathbb{R})$ via fractional transformations.

$\Gamma = SL_2(\mathbb{Z})$.

$\Gamma(m) = \{\gamma \in \Gamma \mid \gamma \equiv 1_2 \pmod{m}\}$ the principal congruence subgroup of level m , $m \in \mathbb{Z}_+$.

$\mathbb{E}_{a,b}$ = the 1-dimensional torus $\mathbb{C}/(\mathbb{Z}a + \mathbb{Z}b)$ for two \mathbb{R} -independent complex numbers a, b .

$\mathbb{E}_{a,b}(d)$ = the d -torsion of $\mathbb{E}_{a,b}$ for a positive integer d .

2 Preliminary

Here we recall the definitions of Heisenberg group and theta functions, and list some of their basic properties that will be used in the context of this paper. For the details, we refer the readers to some standard text books on theta functions, e.g. [11].

Definition .

(i). $\mathbb{G} = \mathbb{C}_1^* \times \mathbb{R} \times \mathbb{R}$, ($\mathbb{C}_1^* = \{\alpha \in \mathbb{C}^* \mid |\alpha| = 1\}$) : the (3-dimensional) Heisenberg group with the group law :

$$(\alpha, \delta, \mu) \cdot (\alpha', \delta', \mu') = (\alpha\alpha' e^{2\pi i \mu \delta'}, \delta + \delta', \mu + \mu') .$$

(ii). For $p, q \in \mathbb{Q} - \{0\}$, $\Lambda(p, q) =$ the subgroup of \mathbb{G} generated by $(1, p, 0), (1, 0, q)$.

(iii). $\mathbb{G}_d = \nu_d \times \mathbb{Z}_d \times \mathbb{Z}_d$, ($\nu_d = \{\alpha \in \mathbb{C}^* \mid \alpha^d = 1\}$) : the finite Heisenberg group of degree d with the group law

$$(\alpha, \bar{\delta}, \bar{\mu}) \cdot (\alpha', \bar{\delta}', \bar{\mu}') = (\alpha\alpha' e^{2\pi i \mu \delta'}, \bar{\delta} + \bar{\delta}', \bar{\mu} + \bar{\mu}').$$

(iv). $\tilde{\mathbb{G}}_d = \mathbb{G}_d \cdot \mathbb{Z}_2$: the extended degree d Heisenberg group which is the semidirect product of \mathbb{G}_d and \mathbb{Z}_2 with \mathbb{G}_d normal in $\tilde{\mathbb{G}}_d$, where the conjugate action on \mathbb{G}_d of the non-trivial element of \mathbb{Z}_2 is given by

$$(\alpha, \bar{\delta}, \bar{\mu}) \mapsto (\alpha, -\bar{\delta}, -\bar{\mu}) , \text{ for } (\alpha, \bar{\delta}, \bar{\mu}) \in \mathbb{G}_d .$$

(v). The canonical representaton of $\tilde{\mathbb{G}}_d$ is by definition the d -dimensional irreducible representation of $\tilde{\mathbb{G}}_d$ where group elements act on a basis $\{e_k\}_{k=0}^{d-1}$ of the vector space by

$$\begin{aligned} ((\alpha, 0, 0) \times 0)e_k &= \alpha e_k, \\ ((1, \bar{1}, 0) \times 0)e_k &= e_{k+1}, \quad ((1, 0, \bar{1}) \times 0)e_k = e^{2\pi i \frac{k}{d}} e_k, \\ ((1, 0, 0) \times \bar{1})e_k &= e_{d-k} , \quad \text{for } 0 \leq k \leq d-1, \quad e_d := e_0 . \end{aligned} \quad (2)$$

□

One has the exact sequence of groups:

$$1 \longrightarrow \Lambda(1, d) \longrightarrow \Lambda\left(\frac{1}{d}, 1\right) \xrightarrow{\phi} \mathbb{G}_d \longrightarrow 1 , \quad (3)$$

where ϕ is the group homomorphism with

$$\phi\left(1, \frac{1}{d}, 0\right) = (1, \bar{1}, 0), \quad \phi(1, 0, 1) = (1, 0, \bar{1}).$$

For $\delta, \mu \in \mathbb{R}, \tau \in \mathbb{H}$ and an entire function f on \mathbb{C} , one defines the functions $S_\mu f$ and $T_\delta(\tau) f$ by

$$\begin{aligned} (S_\mu f)(z) &= f(z + \mu), \\ (T_\delta(\tau) f)(z) &= q^{\delta^2} e^{2\pi i \delta z} f(z + \delta\tau), \quad \text{for } z \in \mathbb{C}. \end{aligned}$$

Then S_μ and $T_\delta(\tau)$ acts on the space of entire functions with the relations,

$$S_\mu S_{\mu'} = S_{\mu+\mu'} , \quad T_\delta(\tau) T_{\delta'}(\tau) = T_{\delta+\delta'}(\tau) , \quad S_\mu T_\delta(\tau) = e^{2\pi i \mu \delta} T_\delta(\tau) S_\mu , \quad (4)$$

and they generate a representation of the Heisenberg group \mathbb{G} by

$$(\alpha, \delta, \mu) \circ f := \alpha T_\delta(\tau) S_\mu f , \quad \text{for } (\alpha, \delta, \mu) \in \mathbb{G} , \quad f : \mathbb{C} \rightarrow \mathbb{C} .$$

We shall also write T_δ instead of $T_\delta(\tau)$ when no confusion arises. For $\tau \in \mathbb{H}$, the the theta function $\vartheta(z, \tau)$ of $\mathbb{E}_{\tau, 1}$ and the theta function $\vartheta[\frac{\delta}{\mu}](z, \tau)$ with characteristics $\delta, \mu \in \mathbb{R}$ are defined by

$$\begin{aligned} \vartheta(z) \quad (= \vartheta(z, \tau)) &:= \sum_{m \in \mathbb{Z}} (T_n 1)(z) = \sum_{m=\infty}^{\infty} q^{m^2} e^{2\pi i m z} , \\ \vartheta\left[\frac{\delta}{\mu}\right](z, \tau) &:= S_\mu T_\delta \vartheta(z) = q^{\delta^2} e^{2\pi i \delta(z+\mu)} \vartheta(z + \delta\tau + \mu), \quad z \in \mathbb{C}, \end{aligned}$$

where 1 is the function with constant value one. Then $\vartheta(z, \tau)$ is the unique entire function invariant under $\Lambda(1, 1)$, and we have

$$T_\alpha \vartheta\left[\frac{\delta}{\mu}\right] = e^{-2\pi i \alpha \mu} \vartheta\left[\frac{\delta + \alpha}{\mu}\right], \quad S_\beta \vartheta\left[\frac{\delta}{\mu}\right] = \vartheta\left[\frac{\delta}{\mu + \beta}\right]. \quad (5)$$

It is known that the theta function has the following representation of infinite product:

$$\vartheta(z, \tau) = \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2\pi z + q^{4n-2})(1 - q^{2n}), \quad q := e^{\pi i \tau},$$

and satisfies the quasi-periodicity and evenness relations:

$$\begin{aligned} \vartheta(z+1) &= \vartheta(z), \\ \vartheta(z+\tau) &= q^{-1} e^{-2\pi i z} \vartheta(z), \\ \vartheta(-z) &= \vartheta(z). \end{aligned}$$

Note that the variable \mathbf{q} in Sect. 1 relates to the above q by

$$\mathbf{q} = q^2.$$

The following relations hold for $\vartheta\left[\frac{\delta}{\mu}\right]$:

$$\begin{aligned} \vartheta\left[\frac{\delta}{\mu}\right](z+1, \tau) &= e^{2\pi i \delta} \vartheta\left[\frac{\delta}{\mu}\right](z, \tau), \\ \vartheta\left[\frac{\delta}{\mu}\right](z+\tau, \tau) &= q^{-1} e^{-2\pi i (z+\mu)} \vartheta\left[\frac{\delta}{\mu}\right](z, \tau), \\ \vartheta\left[\frac{\delta}{\mu}\right](z, \tau) = 0 &\iff z \equiv (\frac{1}{2} - \delta)\tau + (\frac{1}{2} - \mu) \pmod{\mathbb{Z}\tau + \mathbb{Z}}. \end{aligned} \quad (6)$$

We have

$$\begin{aligned} \vartheta\left[\frac{\delta' + \delta}{\mu' + \mu}\right](z, \tau) &= q^{\delta^2} e^{2\pi i \delta(z+\mu+\mu')} \vartheta\left[\frac{\delta'}{\mu'}\right](z + \delta\tau + \mu, \tau) \\ \vartheta\left[\frac{\delta + 1}{\mu}\right](z, \tau) &= \vartheta\left[\frac{\delta}{\mu}\right](z, \tau), \\ \vartheta\left[\frac{\delta}{\mu + 1}\right](z, \tau) &= e^{2\pi i \delta} \vartheta\left[\frac{\delta}{\mu}\right](z, \tau). \end{aligned} \quad (7)$$

Hence

$$\begin{aligned} \vartheta\left[\frac{\delta}{\mu}\right](-z, \tau) &= \vartheta\left[\frac{1-\delta}{-\mu}\right](z, \tau) = e^{-2\pi i \delta} \vartheta\left[\frac{1-\delta}{1-\mu}\right](z, \tau) \\ \vartheta\left[\frac{1-\delta}{1-\mu}\right](\frac{\tau+1}{2}, \tau) &= -e^{2\pi i \mu} \vartheta\left[\frac{\delta}{\mu}\right](\frac{\tau+1}{2}, \tau). \end{aligned} \quad (8)$$

The infinite product representation of four theta functions with half-integer characteristics are given by

$$\begin{aligned} \vartheta_1(z, \tau) &:= \vartheta\left[\frac{\frac{1}{2}}{\frac{1}{2}}\right](z, \tau) = 2q_0 q^{\frac{1}{4}} \sin \pi z \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos 2\pi z + q^{4n}), \\ \vartheta_2(z, \tau) &:= \vartheta\left[\frac{\frac{1}{2}}{0}\right](z, \tau) = 2q_0 q^{\frac{1}{4}} \cos \pi z \prod_{n=1}^{\infty} (1 + 2q^{2n} \cos 2\pi z + q^{4n}), \\ \vartheta_3(z, \tau) &:= \vartheta\left[\frac{0}{0}\right](z, \tau) = q_0 \prod_{n=1}^{\infty} (1 + 2q^{2n-1} \cos 2\pi z + q^{4n-2}), \\ \vartheta_4(z, \tau) &:= \vartheta\left[\frac{0}{\frac{1}{2}}\right](z, \tau) = q_0 \prod_{n=1}^{\infty} (1 - 2q^{2n-1} \cos 2\pi z + q^{4n-2}), \end{aligned} \quad (9)$$

where $q_0 := \prod_{n=1}^{\infty} (1 - q^{2n})$. The functions $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ have zeros at $z = m + n\tau, m + \frac{1}{2} + n\tau, m + \frac{1}{2} + (n + \frac{1}{2})\tau, m + (n + \frac{1}{2})\tau$ respectively for integers m and n , with ϑ_1 odd function and $\vartheta_2, \vartheta_3, \vartheta_4$ even functions of the variable z . The above theta functions satisfy the square relations:

$$\begin{aligned}\vartheta_1^2(z, \tau)\vartheta_2^2(0, \tau) &= \vartheta_4^2(z, \tau)\vartheta_3^2(0, \tau) - \vartheta_3^2(z, \tau)\vartheta_4^2(0, \tau), \\ \vartheta_1^2(z, \tau)\vartheta_3^2(0, \tau) &= \vartheta_4^2(z, \tau)\vartheta_2^2(0, \tau) - \vartheta_2^2(z, \tau)\vartheta_4^2(0, \tau), \\ \vartheta_1^2(z, \tau)\vartheta_4^2(0, \tau) &= \vartheta_3^2(z, \tau)\vartheta_2^2(0, \tau) - \vartheta_2^2(z, \tau)\vartheta_3^2(0, \tau), \\ \vartheta_4^2(z, \tau)\vartheta_4^2(0, \tau) &= \vartheta_3^2(z, \tau)\vartheta_3^2(0, \tau) - \vartheta_2^2(z, \tau)\vartheta_2^2(0, \tau),\end{aligned}\tag{10}$$

hence the identity of the theta functions of zero argument:

$$\vartheta_2^4(0, \tau) + \vartheta_4^4(0, \tau) = \vartheta_3^4(0, \tau).\tag{11}$$

For a positive integer d , let $\text{Th}_d(\tau)$ be the space of theta functions with characteristics which are invariant under the group $\Lambda(1, d)$. Then $\text{Th}_d(\tau)$ is a d -dimensional vector space with the basis

$$\vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right] = \vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right](z, \tau), \quad k = 0, \dots, d-1.$$

By (5) (7), one has

$$T_{\frac{1}{d}}\vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right] = \vartheta\left[\begin{array}{c} \frac{k+1}{d} \\ 0 \end{array}\right], \quad S_1\vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right] = e^{\frac{2k\pi i}{d}}\vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right].$$

By (2) (3) and (8), the action of $T_{\frac{1}{d}}$ and S_1 on $\text{Th}_d(\tau)$, together with the involution,

$$\vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right](z) \mapsto \vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right](-z),$$

gives rise to the canonical representation of the extended degree d Heisenberg group $\tilde{\mathbb{G}}_d$ via the identification:

$$e_k = \vartheta\left[\begin{array}{c} \frac{k}{d} \\ 0 \end{array}\right] \text{ for } 0 \leq k \leq d.$$

3 ODEs for Mirror Map of Elliptic Curves

In this section, we shall characterize the differential operators in Table (I) through J -function with an emphasis on the property of integral \mathbf{q} -expansion of the variable z . It is known that an elliptic curve can be represented in the Weierstrass form:

$$y^2 = 4x^3 - g_2x - g_3, \quad (x, y) \in \mathbb{C}^2, \tag{12}$$

with the parameter

$$[g_2, g_3] \in \mathbb{P}_{(2,3)}^1,$$

equivalently the value

$$J := \frac{g_2^3}{g_2^3 - 27g_3^2}$$

in $\mathbb{C} \cup \{\infty\}$, which is isomorphic to the Riemann surface $\Gamma \backslash \mathbb{H}$ via the modular function $J(\tau)$ of level 1. The periods of elliptic curves satisfy the Picard-Fuchs equation:

$$\frac{d^2y}{dJ^2} + \frac{1}{J} \frac{dy}{dJ} + \frac{31J - 4}{144J^2(1 - J)^2}y = 0,$$

hence are expressed by the Riemann P -function:

$$P \left\{ \begin{array}{cccc} 0 & 1 & \infty & \\ \frac{1}{6} & \frac{1}{4} & 0 & ; J \\ \frac{-1}{6} & \frac{3}{4} & 0 & \end{array} \right\} .$$

The ratio of two periods gives the variable τ of \mathbb{H} , which as a function of J , satisfies the well-known Schwarzian equation:

$$\{\tau, J\} = \frac{3}{8(1-J)} + \frac{4}{9J^2} + \frac{23}{72J(1-J)} .$$

Here the Schwarzian differential operator on the left hand side is defined by:

$$\{y, x\} = \frac{y'''}{y'} - \frac{3}{2} \left(\frac{y''}{y'} \right)^2 .$$

The inverse function $J(\tau)$ of $\tau(J)$ is a modular function [7] [14], expressed by :

$$1728J(\tau) = \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} , \text{ where } g_2(\tau) = 60G_4(\tau) , \quad g_3(\tau) = 140G_6(\tau) , \quad (13)$$

and G_{2k} 's are the Eisenstein series with the Fourier expansion:

$$G_{2k}(\tau) = \frac{2(-1)^k(2\pi)^{2k}}{(2k-1)!} \left\{ \frac{(-1)^k B_{2k}}{4k} + \sum_{n=1}^{\infty} n^{2k-1} \frac{\mathbf{q}^n}{1-\mathbf{q}^n} \right\} , \quad B_{2k} : \text{ the Bernoulli number} .$$

Hence $1728J(\tau)$ admits an integral \mathbf{q} -series :

$$1728J(\tau) = \mathbf{q}^{-1} + 744 + 196884\mathbf{q} + 21493760\mathbf{q}^2 + \cdots , \quad \mathbf{q} = e^{2\pi i\tau} .$$

In this section, we shall discuss the integral property of the \mathbf{q} -series in (1) and the relation between J and z in Table (I). We state the following simple lemma for later use.

Lemma 1. Let

$$w(z) = z + \sum_{m \geq 2} a_m z^m , \quad z(\mathbf{q}) = \mathbf{q} + \sum_{m \geq 2} k_m \mathbf{q}^m$$

be formal power series with $a_m \in \mathbb{Z}$ for all $m \geq 2$. Then $w(z(\mathbf{q}))$ has an integral \mathbf{q} -expansion if and only if all the k_m 's are integers. \square

Consider the following ordinary differential equations of Fuchsian type:

$$(\Theta^2 - \lambda z(\Theta + \alpha)(\Theta + \beta))y(z) = 0 , \quad (14)$$

where $\Theta = z \frac{d}{dz}$ and λ, α, β are positive rational numbers with

$$\alpha + \beta = 1 , \quad \alpha > \beta .$$

The equation (14) is invariant under the change of variables

$$z \mapsto -z + \frac{1}{\lambda} ,$$

with three regular singular points,

$$z = 0, \frac{1}{\lambda}, \infty .$$

Its solutions are expressed by the Riemann P -function:

$$P \left\{ \begin{array}{ccc} 0 & \frac{1}{\lambda} & \infty \\ 0 & 0 & \alpha \\ 0 & 0 & \beta \end{array} ; z \right\} .$$

By the change of variables, $x = \lambda z$, (14) becomes the hypergeometric equation:

$$x(1-x) \frac{d^2y}{dx^2} + (1-2x) \frac{dy}{dx} - \alpha\beta y = 0 , \quad (15)$$

whose fundamental solutions at $x = 0$ are given by the hypergeometric series

$$y_1(x) = F(\alpha, \beta; 1; x) ,$$

together with another solution, uniquely determined by the form

$$y_2(x) = \log(x)F(\alpha, \beta; 1; x) + \sum_{n=1}^{\infty} a_n x^n .$$

The local system for the equation (15) is described by analytic continuations of $y_1(x)$ and $y_2(x)$, or of any other fundamental solutions:

$$ay_1(x) + by_2(x) \quad \text{and} \quad cy_1(x) + dy_2(x) , \quad \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{C}) .$$

The ratio

$$t(x) = \frac{ay_1(x) + by_2(x)}{ay_1(x) + by_2(x)} ,$$

is invariant under the substitution

$$y(x) \mapsto g(x)y(x)$$

for an arbitrary given function $g(x)$, which transfers the equation (15) into another second order linear differential equation. By choosing

$$g(x) = x^{\frac{-1}{2}}(1-x)^{\frac{-1}{2}} ,$$

the equation is put into the form,

$$\frac{d^2y}{dx^2} + Q(x)y = 0 , \quad \text{with} \quad Q(x) = \frac{1-4\alpha\beta x(1-x)}{4x^2(1-x)^2} .$$

Eliminating y in the system of equations:

$$\left\{ \begin{array}{l} \left(\frac{d^2}{dx^2} + Q(x) \right) y = 0 \\ \left(\frac{d^2}{dx^2} + Q(x) \right) (ty) = 0 \end{array} \right.$$

one obtains the non-linear Schwarzian differential equation for $t(x)$:

$$\{t, x\} = 2Q(x) , \quad (16)$$

whose solutions, known as Schwarz triangle functions, are all equivalent under the action of $SL_2(\mathbb{C})$:

$$t \mapsto \frac{at+b}{ct+d} , \quad \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{C}) .$$

Each solution gives rise to a local uniformization of the punctured disc near $x = 0$. It determines the element

$$t(0) := \lim_{x \rightarrow 0} t(x) \in \mathbb{C} \cup \{\infty\} ,$$

and a parabolic transformation fixing $t(0)$, which is given by the local monodromy around $x = 0$. Therefore the solutions of equation (14) is expressed by

$$P \left\{ \begin{array}{ccc} 0 & \frac{1}{\lambda} & \infty \\ 0 & 0 & \alpha \\ 0 & 0 & \beta \end{array} ; z \right\} = A f_1(z) + B f_2(z) , \quad A, B \in \mathbb{C} ,$$

with

$$\begin{aligned} f_1(z) &= y_1(\lambda z) &= F(\alpha, \beta; 1; \lambda z) , \\ f_2(z) &= y_2(\lambda z) - \log(\lambda) y_1(\lambda z) &= \log(z) f_1(z) + \sum_{n=1}^{\infty} d_n z^n . \end{aligned}$$

The ratio

$$\mathbf{t}(z) = \frac{f_2(z)}{2\pi i f_1(z)} \quad (17)$$

forms an uniformizing coordinate of the punctured disc at $z = 0$, characterized as the solution of the equation

$$\{t, z\} = 2\lambda^2 Q(\lambda z) , \quad (18)$$

satisfying the conditions:

$$\lim_{z \rightarrow 0} \mathbf{t}(z) = \infty , \quad \lim_{\theta \rightarrow 2\pi^-} \mathbf{t}(e^{i\theta} z) = \mathbf{t}(z) + 1 , \quad \lim_{z \rightarrow 0} \frac{e^{2\pi i t}}{z} = 1 . \quad (19)$$

Denote

$$\mathbf{q} = e^{2\pi i \mathbf{t}} .$$

We have an local isomorphism between the z -plane and \mathbf{q} -plane with the relation:

$$z = \mathbf{q} + \sum_{n \geq 2} k_n \mathbf{q}^n , \quad k_n \in \mathbb{C} . \quad (20)$$

Note that the coefficients k_n depend on the data λ, α analytically. The characterization of the equations of type (14) with integral coefficients for its assocoated series (20), i.e.

$$k_n \in \mathbb{Z} \quad \forall n ,$$

will be our main concern in what follows.

The analytical continuation of $\mathbf{t}(z)$ gives rise to a Riemann surface \mathfrak{R} , which is an infinite cover over z -plane outside $\{0, \frac{1}{\lambda}, \infty\}$. By the non-zero Wronskian determinant of the equation (14), one has $\frac{dt}{dz} \neq 0$. The projection of \mathfrak{R} to the \mathbf{t} -plane is a local isomorphism, hence its image $\mathbf{t}(\mathfrak{R})$ forms a domain in \mathbb{P}^1 . We have the following relations between Riemann surfaces:

$$\begin{array}{ccc} \mathfrak{R} & \xrightarrow{\mathbf{t}} & \mathbf{t}(\mathfrak{R}) \subset \mathbb{P}^1 \\ z \downarrow & & \\ \mathbb{P}^1 - \{0, \frac{1}{\lambda}, \infty\} & & \end{array} \quad (21)$$

One can extend \mathfrak{R} to a Riemann surface over the zero-value of z as follows. Since the fundamental solutions of the equation (14) near $z = \infty$ can take the form:

$$z^{-\alpha} p_\alpha\left(\frac{1}{z}\right) , \quad z^{-\beta} p_\beta\left(\frac{1}{z}\right) , \quad |z| \gg 0 ,$$

for p_α and p_β power series in $\frac{1}{z}$ with the constant term 1, on a connected region of \mathfrak{R} near $z = \infty$, one has

$$\mathbf{t}(z) = \frac{az^{-\alpha}p_\alpha(\frac{1}{z}) + bz^{-\beta}p_\beta(\frac{1}{z})}{cz^{-\alpha}p_\alpha(\frac{1}{z}) + dz^{-\beta}p_\beta(\frac{1}{z})}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}), \quad \text{for } |z| \gg 0. \quad (22)$$

Therefore

$$\lim_{z \rightarrow \infty} \mathbf{t}(z) = \frac{b}{d} \in \mathbb{P}^1.$$

Write

$$\alpha - \beta = \frac{l}{k}$$

with k and l two relatively prime positive integers, hence $k \geq 2$. By the expression of \mathbf{t} in (22), there exist some local coordinates w near $z = \infty$, and \tilde{t} near $\mathbf{t} = \frac{b}{d}$ such that over a small punctured disc near $z = \infty$, a connected region of \mathfrak{R} is described by the relation:

$$w^l = \tilde{t}^k, \quad (w, \tilde{t}) \neq (0, 0).$$

Let s be the local coordinate for the desingularization of the above equation. The description of (21) on a small connected region over a disc near $z = \infty$ is now equivalent to the following diagram:

$$\begin{array}{ccc} \{0 < |s| < \epsilon\} & \longrightarrow & \{0 < |\tilde{t}| < \delta\}, \quad s \mapsto \tilde{t} = s^l \\ \downarrow & & \downarrow \\ \{0 < |w| < \delta'\} & & , \quad w = s^k \end{array} \quad (23)$$

hence we have

$$\lim_{\theta \rightarrow 2\pi^-} \tilde{t}(ze^{i\theta}) = e^{2\pi i(\beta-\alpha)} \tilde{t}(z). \quad (24)$$

Extending (23) to the following data,

$$\begin{array}{ccc} \{|s| < \epsilon\} & \longrightarrow & \{|\tilde{t}| < \delta\}, \quad s \mapsto \tilde{t} = s^l \\ \downarrow & & \downarrow \\ \{|w| < \delta'\} & & , \quad w = s^k, \end{array} \quad (25)$$

one obtains a Riemann surface $\overline{\mathfrak{R}}$ as a partial compactification of \mathfrak{R} with the extended diagram of (21):

$$\begin{array}{ccc} \overline{\mathfrak{R}} & \xrightarrow{\mathbf{t}} & \mathbf{t}(\overline{\mathfrak{R}}) \subset \mathbb{P}^1 \\ z \downarrow & & \\ \mathbb{P}^1 - \{0, \frac{1}{\lambda}\} & & . \end{array}$$

Note that the map z branches at $z^{-1}(\infty)$ with the multiplicity k . On the other hand, the (multi-valued) function $\tau(J)$, $J \neq 0, 1$, defines a Riemann surface \mathfrak{R}_J with its partial compactification $\overline{\mathfrak{R}}_J$ as follows:

$$\begin{array}{ccc} \overline{\mathfrak{R}}_J & \xrightarrow{\tau} & \mathbb{H} \subset \mathbb{P}^1 \\ J \downarrow & & \\ \mathbb{C} & & . \end{array}$$

We introduce a notion for the discussion of the relation between \mathbf{t} and τ :

Definition . The Schwarz triangle function $\mathbf{t}(z)$ (17) for the equation (14) is said to be related to J -function if for some morphisms Ψ and Φ , the following diagram commutes:

$$\begin{array}{ccccccc} \mathbb{P}^1 & \supset & \mathbb{P}^1 - \{0, \frac{1}{\lambda}\} & \xleftarrow{z} & \overline{\mathfrak{R}} & \xrightarrow{\mathbf{t}} & \mathbf{t}(\overline{\mathfrak{R}}) \\ \Psi \downarrow & & \downarrow & & \Phi \downarrow & & \parallel \\ \mathbb{P}^1 & \supset & \mathbb{C} & \xleftarrow{J} & \overline{\mathfrak{R}}_J & \xrightarrow{\tau} & \mathbb{H} \subset \mathbb{P}^1 \end{array} \quad (26)$$

□

Theorem 1. All the differential operators of type (14) whose $\mathbf{t}(z)$ is related to J -function and with the integral \mathbf{q} -series $z(\mathbf{q})$ are those listed in Table (I).

The rest of this section will be mainly devoted to the proof of the above theorem. We shall regard a coordinate system of \mathbb{C} as the (affine) coordinate of the Riemann sphere \mathbb{P}^1 via the identification:

$$\mathbb{P}^1 = \mathbb{C} \cup \{\infty\} .$$

As before, by the change of variables $x = \lambda z$, the morphism Ψ in (26) induces the rational map

$$\psi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 , \quad x \mapsto J = \psi(x) .$$

By examining the behavior over critical values of the function $J(\tau)$, the above morphism ψ satisfies the following conditions:

- (i) The critical values of ψ are contained in $\{0, 1, \infty\}$.
- (ii) $\psi^{-1}(\infty) = \{0, 1\}$, and the multiplicity of ψ at 0 is equal to 1, i.e.

$$\text{mult}_\psi(0) = 1 .$$

- (iii) The value $\psi(\infty)$ is equal to 0 or 1. For $x \neq \infty$ with $\psi(x) = 0, 1$, we have

$$\text{mult}_\psi(x) = \begin{cases} 3 & \text{if } \psi(x) = 0, \\ 2 & \text{if } \psi(x) = 1. \end{cases}$$

Lemma 2 . There are exactly 3 solutions for the above ψ :

$$\psi(x) = \begin{cases} \frac{(1+8x)^3}{64x(1-x)^3} , \\ \frac{(1+3x)^3}{27x(1-x)^2} , \\ \frac{1}{4x(1-x)} . \end{cases}$$

Proof. Let d be the degree of the map ψ . By the conditions on ψ , d is greater than or equal to 2, and

$$d = 2 \iff \{\text{critical value of } \psi\} = \{0, 1\} ,$$

in which case, one has $\psi(\infty) = 0$, hence easily see

$$\psi(x) = \frac{1}{4x(1-x)} .$$

Now assume $d \geq 3$. By Hurwitz Theorem, we have

$$2d - 2 = r_0 + r_1 + r_\infty ,$$

where r_j is the sum of ramification indices of elements in $\psi^{-1}(j)$. By (ii), $r_\infty = d - 2$, hence

$$d = r_0 + r_1 .$$

Let k be the multiplicity of ψ at $x = \infty$. By (iii), we have

$$(r_0, r_1) = \begin{cases} (2\frac{d-k}{3} + k - 1, \frac{d}{2}), & \text{if } \psi(\infty) = 0, \\ (2\frac{d}{3}, \frac{d-k}{2} + k - 1), & \text{if } \psi(\infty) = 1. \end{cases}$$

This implies that either

$$d = 4, k = 1, \psi(x) = \frac{a(x+b)^3}{x(1-x)^3},$$

or

$$d = 3, k = 1, \psi(x) = \frac{(x+b)^3}{x(1-x)^2},$$

for some complex numbers $a \neq 0$, and $b \neq 0, -1$. For $d = 3$, there is only one critical point $x \in \mathbb{P}^1 - \{0, 1, \infty, -b\}$, which is given by $x = \frac{b}{3b+2}$. We have

$$\psi\left(\frac{b}{3b+2}\right) = 1,$$

which implies

$$0 = 27b^3 + 27b^2 - 4 = (3b-1)(3b+2)^2, \quad b = \frac{1}{3}.$$

Therefore

$$\psi(x) = \frac{(1+3x)^3}{27x(1-x)^2}.$$

For $d = 4$, there are exactly two critical points x_1, x_2 not in $\{0, 1, \infty, -b\}$, and they satisfy the following properties:

$$\text{mult}_\psi(x_i) = 2, \quad \psi(x_i) = 1 \quad \text{for } i = 1, 2.$$

By the expression of $\psi(x)$, one can easily see that x_i 's are the solutions of

$$x^2 + (2+4b)x - b = 0,$$

whose discriminant is equal to

$$16b^2 + 20b + 4 = 4(4b+1)(b+1) \neq 0.$$

Then the following relations hold for $x = x_1, x_2$:

$$\begin{aligned} x + b &= \frac{3x(1-x)}{1-4x}, & x^3 &= -b(2+4b) + (b+(2+4b)^2)x, \\ \psi(x) &= 27a \frac{x^2}{(1-4x)^3} = 27a \frac{-(2+4b)x+b}{-(108+256b+64(2+4b)^2)x+1+48b+64(2+4b)b}. \end{aligned}$$

Since $\psi(x_1) = \psi(x_2) = 1$ and $x_1 \neq x_2$, this implies

$$0 = \begin{vmatrix} -2-4b & b \\ -108-256b-64(2+4b)^2 & 1+48b+64(2+4b)b \end{vmatrix} = 2(8b-1)(4b+1),$$

hence

$$b = \frac{1}{8}, \quad x_i = \frac{-5 \pm \sqrt{27}}{4},$$

and

$$\psi(x) = \frac{(1+8x)^3}{64x(1-x)^3}.$$

□

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. By Lemma 2, the map Ψ in (26) and the corresponding value of $\alpha - \beta$ ($= 2\alpha - 1$) in (24) are expressed by

$$1728J = 1728\Psi(z) = \begin{cases} C \frac{(1+8\lambda z)^3}{z(1-\lambda z)^3}, & C = \frac{1728}{64\lambda}, \quad \alpha - \beta = \frac{1}{3}; \\ C \frac{(1+3\lambda z)^3}{z(1-\lambda z)^2}, & C = \frac{1728}{27\lambda}, \quad \alpha - \beta = \frac{1}{2}; \\ C \frac{1}{z(1-\lambda z)}, & C = \frac{1728}{4\lambda}, \quad \alpha - \beta = \frac{2}{3}. \end{cases}$$

Since both $1728J$ and z have the integral \mathbf{q} -expansions by Lemma 1, C is equal to 1. Hence

$$1728J = 1728\Psi(z) = \begin{cases} \frac{(1+216z)^3}{z(1-27z)^3}, & \lambda = 27, \quad \alpha - \beta = \frac{1}{3}; \\ \frac{(1+192z)^3}{z(1-64z)^2}, & \lambda = 64, \quad \alpha - \beta = \frac{1}{2}; \\ \frac{1}{z(1-432z)}, & \lambda = 432, \quad \alpha - \beta = \frac{2}{3}, \end{cases} \quad (27)$$

and the corresponding operators in (14) are given by

$$\begin{aligned} \lambda = 27, \quad (\alpha, \beta) &= \left(\frac{2}{3}, \frac{1}{3}\right), \quad \Theta^2 - 3(3\Theta + 2)(3\Theta + 1); \\ \lambda = 64, \quad (\alpha, \beta) &= \left(\frac{3}{4}, \frac{1}{4}\right), \quad \Theta^2 - 4(4\Theta + 3)(4\Theta + 1); \\ \lambda = 432, \quad (\alpha, \beta) &= \left(\frac{5}{6}, \frac{1}{6}\right), \quad \Theta^2 - 12(6\Theta + 5)(6\Theta + 1). \end{aligned}$$

Therefore the above differential operators are the only possibilities satisfying the conditions of Theorem 1. Now we are going to show that the function $\mathbf{t}(z)$ associated to the differential operators in Table (I) does arise from J -function with the corresponding expression of $1728J(z)$ given there, which implies the integral \mathbf{q} -series for $z(\mathbf{q})$ by Lemma 1. Associated to each family of the weighted hypersurfaces in Table (I), there corresponds an algebraic surface, denoted again by:

$$\begin{aligned} P_8 : \quad &x_1^3 + x_2^3 + x_3^3 - sx_1x_2x_3 = 0, \quad ([x_1, x_2, x_3], [1, s]) \in \mathbb{P}^2 \times \mathbb{P}^1; \\ X_8 : \quad &x_1^4 + x_2^4 + x_3^2 - sx_1x_2x_3 = 0, \quad ([x_1, x_2, x_3], [1, s]) \in \mathbb{P}_{(1,1,2)}^2 \times \mathbb{P}^1; \\ J_{10} : \quad &x_1^6 + x_2^3 + x_3^2 - sx_1x_2x_3 = 0, \quad ([x_1, x_2, x_3], [1, s]) \in \mathbb{P}_{(1,2,3)}^2 \times \mathbb{P}^1. \end{aligned}$$

Their singularities are given by

$$\begin{aligned} \text{Sing}(P_8) &= \emptyset; \\ \text{Sing}(X_8) &= \{([0, 0, 1], [1, \infty])\}; \\ \text{Sing}(J_{10}) &= \{([0, 1, 0], [1, \infty]), ([0, 0, 1], [1, \infty])\}. \end{aligned}$$

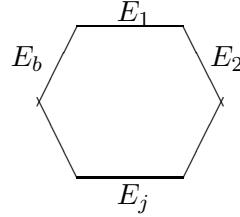
Let

$$S = S(P_8), \quad S(X_8), \quad S(J_{10})$$

be the minimal resolution of the corresponding surface, which is an elliptic surface over \mathbb{P}^1 via the s -projection:

$$\sigma : S \longrightarrow \mathbb{P}^1,$$

with the singular fiber $\sigma^{-1}(\infty)$ of type ${}_1\mathrm{I}_b$:



where

$$b = \begin{cases} 3, & S = S(P_8); \\ 4, & S = S(X_9); \\ 6, & S = S(J_{10}). \end{cases}$$

Let $\{\Gamma_1, \Gamma_2\}$ be a canonical basis of $H_1(\sigma^{-1}(s), \mathbb{Z})$ for $|s| \gg 0$, which is defined by

$$\begin{aligned} \Gamma_1 &= \text{the vanishing circle near a double point of } \sigma^{-1}(\infty), \\ \Gamma_2 &= \text{the invariant circle near } \sigma^{-1}(\infty). \end{aligned}$$

The Picard-Lefschetz transformation along $se^{i\theta}$ as θ varies from 0 to -2π is given by

$$(\Gamma_1, \Gamma_2) \mapsto (\Gamma_1, \Gamma_2) \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

Since the morphism

$$([x_1, x_2, x_3], [1, s]) \mapsto ([e^{2\pi i/b} x_1, x_2, x_3], [1, e^{-2\pi i/b} s]),$$

induces an order b automorphism of the surface S , the period map

$$s \mapsto \left(\int_{\Gamma_2} \omega_s \right) / \left(\int_{\Gamma_1} \omega_s \right), \quad \omega_s = \text{the holomorphic differential of } \sigma^{-1}(s),$$

is determined by the variable $z(:= s^{-b})$. Hence we obtain a (multi-valued) function $\mathbf{t}(z)$ from z -plane to the upper half-plane \mathbb{H} , which is the solution of corresponding differential equation in Table (I), (for its derivation , see e.g. [8]). As z varies along the path $ze^{i\theta}$ from $\theta = 0$ to $\theta = 2\pi$, the change of homology of the fiber in S is described by

$$(\Gamma_1, \Gamma_2) \mapsto (\Gamma_1, \Gamma_2) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore $\mathbf{t}(z)$ satisfies the condition (19). The function $\mathbf{t}(z)$ describes the periods of elliptic fibers of the surface S , hence arises from J -function and we have the diagram (26). By Lemma 2 and (27), we obtain the expression of $1728J(z)$. This completes the proof of Theorem 1. \square

From the relation of variables J and z in Table (I), the map \mathbf{t} in (26) is bijective on the region $\text{Im}(\mathbf{t}) \gg 0$. From (25) and the values of $\alpha - \beta$ in (27), \mathbf{t} is also bijective on a connected region of \mathfrak{R} over $|z| \gg 0$ for the the cases of P_8 and X_9 , but not for J_{10} . For the families of P_8 and X_9 , \mathfrak{R} is indeed isomorphic to the upper-half plane \mathbb{H} via the map \mathbf{t} . In the following sections, we are going to express $z(\mathbf{t})$ in terms of theta functions without appealing to the J -function.

For J_{10} -family, the Riemann surface $\bar{\mathcal{R}}$ is not isomorphic to \mathbb{H} . However, from the relation of J and z , one can easily obtain

$$z(\tau) = \frac{1 - \sqrt{1 - \frac{1}{J(\tau)}}}{864} = \frac{\sqrt{g_2(\tau)^3} - \sqrt{27}g_3(\tau)^2}{864\sqrt{g_2(\tau)^3}}$$

where g_2, g_3 are given by (13). For an elliptic curve in J_{10} -family:

$$(J_{10}) \quad X_s : x_1^6 + x_2^3 + x_3^2 - sx_1x_2x_3 = 0, \quad [x_1, x_2, x_3] \in \mathbb{P}_{(1,2,3)}^2,$$

the substitution,

$$iy = \frac{2x_3 - sx_2x_1}{x_1^3}, \quad x = \frac{x_2 - \frac{s^2}{12}x_1^2}{x_1^2}, \quad (28)$$

changes X_s to the Weierstrass form (12) with

$$g_2 = \frac{s^4}{12}, \quad g_3 = \frac{s^6}{216} - 4.$$

By using the homogeneous coordinates of $\mathbb{P}_{(1,2,3)}^2$,

$$[x_1, x_2, x_3] = [1, x, iy],$$

an elliptic curve of Weierstrass form (12) is expressed by

$$x_3^2 + 4x_3^3 - g_3x_1^6 - g_2x_1^4x_2 = 0, \quad [x_1, x_2, x_3] \in \mathbb{P}_{(1,2,3)}^2,$$

Through Weierstrass function presentation of (12) and the formulae in [16], one obtains the theta function expression of the above x_i 's :

$$\begin{aligned} x_1 &= 2\vartheta_1(z, \tau), \\ x_2 &= \frac{1}{3}(\vartheta_3^4(0, \tau) + \vartheta_3^4(0, \tau))\vartheta_1^2(z, \tau) + \vartheta_3^2(0, \tau)\vartheta_4^2(0, \tau)\vartheta_2^2(z, \tau), \\ x_3 &= -2i\vartheta_2^2(0, \tau)\vartheta_3^2(0, \tau)\vartheta_4^2(0, \tau)\vartheta_2(z, \tau)\vartheta_3(z, \tau)\vartheta_4(z, \tau), \end{aligned}$$

with $J = \frac{g_2^3}{g_2^3 - 27g_3^2}$ given by (13). In a similar way, one can obtain the theta function representation for J_{10} -family via (28).

4 Theta Function Parametrization for P_8 -family

In this section we describe the elliptic theta function representation of elliptic curves in P_8 -family,

$$(P_8) \quad X_s : f_s(x) = x_1^3 + x_2^3 + x_3^3 - sx_1x_2x_3 = 0, \quad [x_1, x_2, x_3] \in \mathbb{P}^2. \quad (29)$$

The moduli parameter z ($:= s^{-3}$) will be expressed by theta constants involved in the representation. First we note that the fundamental locus of the pencil (29) consists of 9 elements:

$$[x_1, x_2, x_3] = [0, -1, \omega^k], [-\omega, 0, -1], [-1, \omega^k, 0], \text{ for } 0 \leq k \leq 2, \quad \omega = e^{\frac{2\pi i}{3}}, \quad (30)$$

each of which gives rise to a section of the elliptic surface:

$$\sigma : S(P_8) \longrightarrow \mathbb{P}^1. \quad (31)$$

Let G be the group of linear transformations preserving the polynomial $f_s(x)$ for a generic s . It is easy to see that the finite group G is generated by the following elements:

$$\begin{aligned} C(x_1, x_2, x_3) &= (\omega x_1, \omega x_2, \omega x_3) , & R(x_1, x_2, x_3) &= (x_1, \omega x_2, \omega^2 x_3) \\ T(x_1, x_2, x_3) &= (x_2, x_3, x_1) , & I(x_1, x_2, x_3) &= (x_1, x_3, x_2) . \end{aligned} \quad (32)$$

The subgroup generated by C, R is described by

$$\langle C, R \rangle = \{ \text{dia.}(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1^3 = \alpha_2^3 = \alpha_3^3 = \alpha_1 \alpha_2 \alpha_3 = 1 \} .$$

By the relation

$$R \cdot T = C (T \cdot R) ,$$

one can easily see that G is isomorphic to the extended degree 3 Heisenberg group $\tilde{\mathbb{G}}_3$:

$$G \cap SL_3(\mathbb{C}) = \langle C, R, T \rangle \simeq \mathbb{G}_3 , \quad G = \langle C, R, T \rangle \cdot \langle I \rangle \simeq \tilde{\mathbb{G}}_3 . \quad (33)$$

The action of G on the homogeneous coordinates x_k 's is now equivalent to the canonical representation of $\tilde{\mathbb{G}}_3$ by identifying x_k with e_k of (2). As a projective transformation group, G acts on \mathbb{P}^2 which leaves each X_s invariant. Let r_s, t_s, ι_s be the automorphisms of X_s induced by R, T, I respectively. Then t_s and r_s generates the group of order 3 translations of X_s , and ι_s is an involution of X_s :

$$\begin{aligned} \langle r_s, t_s \rangle &\simeq \mathbb{Z}_3 \oplus \mathbb{Z}_3 & \simeq \mathbb{G}_3/\text{center}(\mathbb{G}_3) , \\ \langle r_s, t_s, \iota_s \rangle &\simeq (\mathbb{Z}_3 \oplus \mathbb{Z}_3) \cdot \mathbb{Z}_2 & \simeq \tilde{\mathbb{G}}_3/\text{center}(\tilde{\mathbb{G}}_3) . \end{aligned} \quad (34)$$

The fundamental locus (30) is invariant under ι_s with only one fixed element $[0, -1, 1]$, which induces a section of the elliptic surface (31), denoted by

$$\rho : \mathbb{P}^1 \longrightarrow S(P_8) , \quad s \mapsto \rho(s) .$$

All the 9 sections induced by (30) are the translations of the above ρ by elements in $\langle t_s, r_s \rangle$. Denote

$$\mathcal{O}_{X_s}(1) = \text{the restriction of hyperplane bundle on } X_s .$$

By (33) and (34), we have a $\tilde{\mathbb{G}}_3$ -linearization on the line bundle $\mathcal{O}_{X_s}(1)$ via the linear representation of G ,

$$\begin{array}{ccc} \tilde{\mathbb{G}}_3 \times \mathcal{O}_{X_s}(1) & \longrightarrow & \mathcal{O}_{X_s}(1) \\ \downarrow & & \downarrow \\ \tilde{\mathbb{G}}_3 \times X_s & \longrightarrow & X_s . \end{array} \quad (35)$$

We are going to construct this $\tilde{\mathbb{G}}_3$ -linearization from the universal covering space of X_s . It is known that the quotient of \mathbb{P}^2 by $\langle R \rangle$ is a 2-dimensional toric variety :

$$\mathbb{P}^2 / \langle R \rangle \supset \mathbb{C}^{*3} / (\mathbb{C}^{*} \cdot \langle R \rangle) ,$$

and the natural projection defines an equivariant morphism of toric varieties,

$$p : \mathbb{P}^2 \longrightarrow \mathbb{P}^2 / \langle R \rangle .$$

There are 3 toric divisors in $\mathbb{P}^2 / \langle R \rangle$, given by $\xi_i = 0$ for $1 \leq i \leq 3$, here ξ_i 's are sections on $\mathbb{P}^2 / \langle R \rangle$ with $p^*(\xi_i) = x_i$. Note that ξ_i and ξ_j are not linearly equivalent for $i \neq j$, however ξ_i^3 's and $\xi_1 \xi_2 \xi_3$ are sections of the same line bundle. Denote

$$\Xi_s : \xi_1^3 + \xi_2^3 + \xi_3^3 - s \xi_1 \xi_2 \xi_3 = 0 \quad \text{in } \mathbb{P}^2 / \langle R \rangle . \quad (36)$$

The restriction of the projection p defines a 3-fold cover of elliptic curves:

$$p_s : X_s \longrightarrow \Xi_s = X_s / \langle r_s \rangle ,$$

and the automorphisms ι_s, t_s of X_s induce the involution $\iota_{s,0}$ and an order 3 translation $t_{s,0}$ of Ξ_s respectively. The restriction of p_s defines an one-one correspondence between the fixed points ι_s and $\iota_{s,0}$,

$$p_s : X_s^{\iota_s} \simeq \Xi_s^{\iota_{s,0}} ,$$

and denote the element in $\Xi_s^{\iota_{s,0}}$ corresponding to $\rho(s)$ by

$$\rho(s)_0 := p_s(\rho(s)) = \text{zero}(\xi_1).$$

One may regard Ξ_s as an 1-dimensional torus $\mathbb{E}_{\tau,1}$ for some $\tau \in \mathbb{H}$ with

$$\begin{aligned} \iota_{s,0} : \Xi_s &\longrightarrow \Xi_s, & \iff & \iota : \mathbb{E}_{\tau,1} \longrightarrow \mathbb{E}_{\tau,1}, & [z] &\mapsto [-z] \\ t_{s,0} : \Xi_s &\longrightarrow \Xi_s & \iff & t : \mathbb{E}_{\tau,1} \longrightarrow \mathbb{E}_{\tau,1}, & [z] &\mapsto [z + c] , \quad \text{for } [c] \in \mathbb{E}_{\tau,1}(3), \\ \rho(s)_0 \in \Xi_s & & \iff & e \in \mathbb{E}_{\tau,1}(2) . \end{aligned} \quad (37)$$

The above data indeed determine the algebraic form (36) by the following lemma.

Lemma 3 . Let \mathbb{E} be an elliptic curve with an involution ι and an order 3 translation t . Let e be an element of \mathbb{E} fixed by ι . Then:

(i) The following divisors are linearly equivalent:

$$e + t(e) + t^2(e) \sim 3e \sim 3t(e) \sim 3t^2(e).$$

(ii) There exist non-trivial sections f_i in $\Gamma(\mathbb{E}, \mathcal{O}(t^{i-1}(e)))$, $1 \leq i \leq 3$, such that

$$f_1^3 + f_2^3 + f_3^3 = sf_1f_2f_3 \quad \in \Gamma(\mathbb{E}, \mathcal{O}(3e)) ,$$

for some $s \in \mathbb{C} - \{0\}$.

Proof. Since

$$\iota(t(e)) = (\iota t \iota)(e) = t^{-1}(e) = t^2(e),$$

we have

$$2e \sim t(e) + t^2(e) ,$$

hence

$$3e \sim e + t(e) + t^2(e) .$$

Applying t and t^2 to the above relation, we obtain (i). For $1 \leq i \leq 3$, let f_i be a non-trivial element in $\Gamma(\mathbb{E}, \mathcal{O}(t^{i-1}(e)))$. Since $\Gamma(\mathbb{E}, \mathcal{O}(3e))$ is a 3-dimensional vector space with $\{f_i^3\}_{i=1}^3$ as a basis, we have the relation:

$$f_1f_2f_3 = \beta_1f_1^3 + \beta_2f_2^3 + \beta_3f_3^3 ,$$

for $\beta_i \in \mathbb{C}$. As $e, t(e)$ and $t^2(e)$ are distinct elements in \mathbb{E} , one concludes $\beta_i \neq 0$ for all i . Replacing f_i by $\beta_i^{\frac{1}{3}}f_i$, we obtain (ii). \square

One may describe the above f_i 's by theta functions with characteristics on an elliptic curve $\mathbb{E}_{\tau,1}$, and then the equation in (ii) simply means the relation among those theta functions. A such relation is given as follows.

Lemma 4. For $\tau \in \mathbb{H}$, let p_1, p_2, p_3 be the elements in $\mathbb{E}_{\tau,1}$ defined by

$$p_1 = [\frac{\tau}{2} + \frac{1}{2}], \quad p_2 = [\frac{\tau}{6} + \frac{1}{2}], \quad p_3 = [\frac{5\tau}{6} + \frac{1}{2}].$$

Let

$$\xi_1 = \vartheta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](z, \tau), \quad \xi_2 = \vartheta[\begin{smallmatrix} \frac{1}{3} \\ 0 \end{smallmatrix}](z, \tau), \quad \xi_3 = \vartheta[\begin{smallmatrix} \frac{2}{3} \\ 0 \end{smallmatrix}](z, \tau).$$

Then

$$\xi_1^3, \xi_2^3, \xi_3^3, \xi_1 \xi_2 \xi_3 \in \Gamma(\mathbb{E}_{\tau,1}, \mathcal{O}(\sum_{i=1}^3 p_i))$$

and the following relation holds:

$$\xi_1^3 + \xi_2^3 + \xi_3^3 = s \xi_1 \xi_2 \xi_3, \quad (38)$$

with s given by

$$s^{-1} = \frac{q^{5/9} \vartheta(0, \tau) \vartheta(\frac{\tau}{3}, \tau) \vartheta(\frac{2\tau}{3}, \tau)}{\vartheta(0, \tau)^3 + q^{1/3} \vartheta(\frac{\tau}{3}, \tau)^3 + q^{4/3} \vartheta(\frac{2\tau}{3}, \tau)^3}.$$

Proof. By (6), the zeros of $\xi_1 \xi_2 \xi_3$ are $p_1 + p_2 + p_3$, and the functions $\xi_1^3, \xi_2^3, \xi_3^3, \xi_1 \xi_2 \xi_3$ have the same quasi-periodicity condition, hence define four sections in $\Gamma(\mathbb{E}_{\tau,1}, \mathcal{O}(\sum_{i=1}^3 p_i))$. Now the linear dependence of $\xi_1^3 + \xi_2^3 + \xi_3^3$ and $\xi_1 \xi_2 \xi_3$ is equivalent to

$$\text{zero } (\xi_1^3 + \xi_2^3 + \xi_3^3) = p_1 + p_2 + p_3,$$

which will follow from

$$(\xi_2^3 + \xi_3^3)(p_1) = (\xi_1^3 + \xi_3^3)(p_2) = (\xi_1^3 + \xi_2^3)(p_3) = 0.$$

By (6) (8), we have

$$\begin{aligned} \vartheta[\begin{smallmatrix} \frac{2}{3} \\ 0 \end{smallmatrix}](\frac{\tau+1}{2}, \tau) &= -e^{2\pi i/3} \vartheta[\begin{smallmatrix} \frac{1}{3} \\ 0 \end{smallmatrix}](\frac{\tau+1}{2}, \tau); \\ \vartheta[\begin{smallmatrix} \frac{2}{3} \\ 0 \end{smallmatrix}](\frac{\tau+1}{2} - \frac{\tau}{3}, \tau) &= e^{2\pi i \frac{\tau+1}{3}} \vartheta(\frac{\tau+1}{2} + \frac{\tau}{3}) = e^{2\pi i \frac{\tau+1}{3}} \vartheta(-\frac{\tau+1}{2} - \frac{\tau}{3}) \\ &= -e^{2\pi i/3} \vartheta(\frac{\tau+1}{2} - \frac{\tau}{3}); \\ \vartheta[\begin{smallmatrix} \frac{1}{3} \\ 0 \end{smallmatrix}](\frac{\tau+1}{2} - \frac{2\tau}{3}, \tau) &= e^{\pi i/3} \vartheta(\frac{\tau+1}{2} - \frac{\tau}{3}) = e^{\pi i/3} \vartheta(-\frac{\tau+1}{2} + \frac{\tau}{3}) \\ &= e^{\pi i/3} \vartheta(\frac{\tau+1}{2} - \frac{2\tau}{3}). \end{aligned}$$

Therefore we obtain the relation (38), whose value at $z = 0$ gives the expression of s . \square

Remark. By a similar argument, one can also have the relation (38) by setting

$$\begin{aligned} \xi_1 &= \vartheta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](z, \tau), \quad \xi_2 = \vartheta[\begin{smallmatrix} 0 \\ \frac{1}{3} \end{smallmatrix}](z, \tau), \quad \xi_3 = \vartheta[\begin{smallmatrix} 0 \\ \frac{2}{3} \end{smallmatrix}](z, \tau); \\ \xi_1 &= \vartheta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](z, \tau), \quad \xi_2 = e^{8\pi i/9} \vartheta[\begin{smallmatrix} \frac{1}{3} \\ 0 \end{smallmatrix}](z, \tau), \quad \xi_3 = e^{8\pi i/9} \vartheta[\begin{smallmatrix} \frac{2}{3} \\ 0 \end{smallmatrix}](z, \tau); \\ \xi_1 &= \vartheta[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}](z, \tau), \quad \xi_2 = e^{-2\pi i/9} \vartheta[\begin{smallmatrix} \frac{1}{3} \\ \frac{2}{3} \end{smallmatrix}](z, \tau), \quad \xi_3 = e^{-2\pi i/9} \vartheta[\begin{smallmatrix} \frac{2}{3} \\ \frac{1}{3} \end{smallmatrix}](z, \tau); \end{aligned} \quad (39)$$

with the corresponding s given by

$$\begin{aligned} s^{-1} &= \frac{\vartheta(0, \tau)\vartheta(\frac{1}{3}, \tau)\vartheta(\frac{2}{3}, \tau)}{\vartheta(0, \tau)^3 + \vartheta(\frac{1}{3}, \tau)^3 + \vartheta(\frac{2}{3}, \tau)^3} ; \\ s^{-1} &= \frac{q^{5/9}e^{8\pi i/9}\vartheta(0, \tau)\vartheta(\frac{\tau+1}{3}, \tau)\vartheta(\frac{2\tau+2}{3}, \tau)}{\vartheta(0, \tau)^3 + q^{1/3}e^{-2\pi i/3}\vartheta(\frac{\tau+1}{3}, \tau)^3 + q^{4/3}e^{-2\pi i/3}\vartheta(\frac{2\tau+2}{3}, \tau)^3} ; \\ s^{-1} &= \frac{q^{5/9}e^{4\pi i/9}\vartheta(0, \tau)\vartheta(\frac{\tau+2}{3}, \tau)\vartheta(\frac{2\tau+1}{3}, \tau)}{\vartheta(0, \tau)^3 + q^{1/3}e^{2\pi i/3}\vartheta(\frac{\tau+2}{3}, \tau)^3 + q^{4/3}e^{2\pi i/3}\vartheta(\frac{2\tau+1}{3}, \tau)^3} . \end{aligned}$$

With the elliptic curve Ξ_s identified with $\mathbb{E}_{\tau,1}$ as in (37), one can write $X_s = \mathbb{C}/L$ for an index 3 sublattice L of $\mathbb{Z}\tau + \mathbb{Z}$. With the complex number c in (37), one may assume that

$$\mathbb{C} \longrightarrow \mathbb{C}, \quad z \mapsto z + c ,$$

induces the order 3 automorphism t_s of X_s . Then one can easily conclude

$$(X_s, \langle t_s \rangle, \langle r_s \rangle) = \begin{cases} (\mathbb{E}_{\tau,3}, \langle [z] \mapsto [z + \frac{\tau}{3}] \rangle, \langle [z] \mapsto [z + 1] \rangle) & \text{if } c = \pm \frac{\tau}{3}, \\ (\mathbb{E}_{\tau+1,3}, \langle [z] \mapsto [z + \frac{\tau+1}{3}] \rangle, \langle [z] \mapsto [z + 1] \rangle) & \text{if } c = \pm \frac{\tau+1}{3}, \\ (\mathbb{E}_{\tau+2,3}, \langle [z] \mapsto [z + \frac{\tau+2}{3}] \rangle, \langle [z] \mapsto [z + 1] \rangle) & \text{if } c = \pm \frac{\tau+2}{3}, \\ (\mathbb{E}_{3\tau,1}, \langle [z] \mapsto [z + \frac{1}{3}] \rangle, \langle [z] \mapsto [z + \tau] \rangle) & \text{if } c = \pm \frac{1}{3}. \end{cases} \quad (40)$$

Consider ξ_i and x_i as functions on the universal covering space \mathbb{C} of Ξ_s , and regard the fundamental group of Ξ_s as a subgroup of the Heisenberg group \mathbb{G} , which acts on entire functions of \mathbb{C} as in Sect. 2. By (40) and (32), ξ_1 corresponds to the $\Lambda(1,1)$ -entire function on \mathbb{C} , hence we have

$$\xi_1 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z, \tau) .$$

By Lemma 3, renumbering ξ_2 and ξ_3 if necessary, one may represent ξ_i 's by theta functions either in Lemma 4, or those in (39). By (32) (37), one requires

$$\xi_2(-z) = \xi_3(z) ,$$

hence by (8), we have

$$\xi_2(z) = \vartheta \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} (z, \tau) \quad \text{or} \quad \vartheta \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} (z, \tau) .$$

By (32) (40), one has the relation:

$$\xi_2(z+1) = \omega \xi_2(z) ,$$

therefore

$$\xi_2(z) = \vartheta \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} (z, \tau) .$$

Then Lemma 4 implies the following result:

Theorem 2. With x_i, ξ_i ($1 \leq i \leq 3$) the coordinates of elliptic curves X_s, Ξ_s in P_8 -family, p_s the morphism between them, and r_s, t_s, ι_s the automorphisms of X_s as before. For $\tau \in \mathbb{H}$, define $s(\tau)$ by

$$s(\tau)^{-1} = \frac{q^{5/9}\vartheta(0, \tau)\vartheta(\frac{\tau}{3}, \tau)\vartheta(\frac{2\tau}{3}, \tau)}{\vartheta(0, \tau)^3 + q^{1/3}\vartheta(\frac{\tau}{3}, \tau)^3 + q^{4/3}\vartheta(\frac{2\tau}{3}, \tau)^3} .$$

Then the above data for X_s, Ξ_s have the following realization in complex tori:

$$\begin{aligned} X_s &= \mathbb{E}_{\tau,3} , & \Xi_s &= \mathbb{E}_{\tau,1} , & p_s : X_s &\longrightarrow \Xi_s , [z] \mapsto [z] ; \\ \xi_1 &= \vartheta \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right] (z, \tau) , & \xi_2 &= \vartheta \left[\begin{smallmatrix} \frac{1}{3} \\ 0 \end{smallmatrix} \right] (z, \tau) , & \xi_3 &= \vartheta \left[\begin{smallmatrix} \frac{2}{3} \\ 0 \end{smallmatrix} \right] (z, \tau) ; \\ t_s : \mathbb{E}_{\tau,3} &\longrightarrow \mathbb{E}_{\tau,3} , & [z] &\mapsto [z + \frac{\tau}{3}] ; \\ r_s : \mathbb{E}_{\tau,3} &\longrightarrow \mathbb{E}_{\tau,3} , & [z] &\mapsto [z + 1] ; \\ \iota_s : \mathbb{E}_{\tau,3} &\longrightarrow \mathbb{E}_{\tau,3} , & [z] &\mapsto [-z], \end{aligned}$$

and the projective representation of $\langle r_s, t_s, \iota_s \rangle$ on x_i 's is given by the canonical representation of $\tilde{\mathbb{G}}_3$ on $\text{Th}_3(\tau)$. \square

By the above expression of $s(\tau)^{-1}$, we now derive the formula of the variable z ($:= s^{-3}$) as an function of \mathbf{q} in the following theorem.

Theorem 3 . The function $z(\mathbf{q})$ for P_8 -family of Table (I) is given by

$$z(\mathbf{q}) = \frac{\mathbf{q}^{5/2} \vartheta(0, 3\mathbf{t})^3 \vartheta(\mathbf{t}, 3\mathbf{t})^3 \vartheta(2\mathbf{t}, 3\mathbf{t})^3}{(\vartheta(0, 3\mathbf{t})^3 + \mathbf{q}^{1/2} \vartheta(\mathbf{t}, 3\mathbf{t})^3 + \mathbf{q}^2 \vartheta(2\mathbf{t}, 3\mathbf{t})^3)^3}, \quad \mathbf{q} = e^{2\pi i \mathbf{t}},$$

and it has the integral \mathbf{q} -expansion with

$$\lim_{\mathbf{q} \rightarrow 0} \frac{z(\mathbf{q})}{\mathbf{q}} = 1.$$

Proof. By Theorem 2, for $s = s(\tau)$, we have

$$X_s = \mathbb{E}_{\tau,3} \simeq \mathbb{E}_{\mathbf{t},1}, \quad \mathbf{t} = \frac{\tau}{3},$$

then one obtains the above expression of $z(\mathbf{q})$. By the infinite q -product representation of the theta function, the ratio $\frac{z(\mathbf{q})}{\mathbf{q}}$ tends to 1 as $\mathbf{q} \rightarrow 0$, and $z(\mathbf{q})$ is an integral power series of $\sqrt{\mathbf{q}}$. The variable \mathbf{t} is obtained as the ratio of two periods of the holomorphic differential of X_s . As X_s is isomorphic to X_{ω_s} , \mathbf{t} can be considered as a multi-valued function of z . Since the periods satisfy the Fuchsian equation (14) for $\lambda = 27, \alpha = \frac{2}{3}, \beta = \frac{1}{3}$, \mathbf{t} is a solution of the corresponding Schwarzian equation (18). It can be shown that $\mathbf{t}(z)$ satisfies the condition (19), hence \mathbf{t} is the variable in Sect. 3. Since z is a function of \mathbf{q} , this implies $z(\mathbf{q})$ is a power series of \mathbf{q} with integral coefficients expressed in (1). \square

Remark.

- (i) In this expression of $z(\mathbf{q})$, both the numerator and denominator are integral power series of $\sqrt{\mathbf{q}}$, but not in \mathbf{q} . However their ratio gives an integral \mathbf{q} -expansion for z .
- (ii) Since the surface $S(P_8)$ of (31) is the universal family of $(\mathbb{E}, \mathbb{E}(3))$ for 1-torus \mathbb{E} with 3-torsion $\mathbb{E}(3)$, the correspondence

$$\tau \mapsto s^{-1} \left(\frac{\tau}{3} \right),$$

defines an isomorphism between $\Gamma(3) \setminus \mathbb{H}$ and \mathbb{P}^1 .

5 Elliptic Curves in Ising Model

Now we start to investigate the relation between the constrained polynomials of X_9 -family and the Boltzmann weights in Ising model. Let us first recall the Jacobi elliptic function parametrization in Ising model. This theory has been extensively discussed in many literatures, e.g. [2] [3] [15]. Here we adopt the formulation in chiral Potts N -state model models [4] [13], even though prime interests of which were on hyperelliptic curves for $N \geq 3$, however the parametrization works also for $N = 2$, i.e. the case of Ising model.

Let W be an 1-dimensional torus ($= \mathbb{C}/\text{lattice}$), and consider the morphisms of W ,

$$\begin{aligned}\theta : W &\longrightarrow W, \quad [z] \mapsto [-z] \quad ; \\ m : W &\longrightarrow W, \quad [z] \mapsto [z + z_0], \quad [z_0] \in E(2) \quad ; \\ \sigma : W &\longrightarrow W, \quad \sigma = m \cdot \theta \quad .\end{aligned}$$

One can present W as a plane curve through the following commutative diagram:

$$\begin{array}{ccccc} W & \xrightarrow{\Psi} & \mathbb{P}^1 & = W / \langle \theta \rangle \\ \downarrow \Pi & & \downarrow \pi & & \\ W / \langle \sigma \rangle & = & \mathbb{P}^1 & \xrightarrow{\psi} & \mathbb{P}^1 = W / \langle \theta, \sigma \rangle , \end{array} \quad (41)$$

where Ψ, ψ, Π, π are natural projections. For some suitable coordinates of \mathbb{P}^1 , ψ, π have the expressions:

$$\psi(t) = t^2, \quad \pi(\lambda) = \frac{(1 - k'\lambda)(1 - k'\lambda^{-1})}{k^2}, \quad t, \lambda \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\},$$

where k', k are elements in $\mathbb{C} - \{0, \pm 1\}$ satisfying the relation:

$$k^2 + k'^2 = 1.$$

Then W is isomorphic to the algebraic curve:

$$W_{k'} : \quad t^2 = \frac{(1 - k'\lambda)(1 - k'\lambda^{-1})}{k^2}, \quad (t, \lambda) \in \mathbb{C}^2, \quad (42)$$

with

$$\Psi(t, \lambda) = \lambda, \quad \Pi(t, \lambda) = t, \quad \theta(t, \lambda) = (-t, \lambda), \quad \sigma(t, \lambda) = (t, \lambda^{-1}).$$

The branched data of Ψ and Π are given by

$$\begin{aligned}\text{Branched points of } \Psi : \quad & p = (\infty, 0), \quad p' = (\infty, \infty), \quad q = (0, k'), \quad q' = (0, k'^{-1}) \\ \text{Branched points of } \Pi : \quad & b_{\pm} = (\pm \sqrt{\frac{1-k'}{1+k'}}, 1), \quad b'_{\pm} = (\pm \sqrt{\frac{1+k'}{1-k'}}, -1).\end{aligned} \quad (43)$$

By the transformations,

$$w = \frac{k'}{k^2}(\lambda - \frac{1}{\lambda}), \quad \lambda = \frac{1}{2k'}\{k^2(w - t^2) + k'^2 + 1\},$$

$W_{k'}$ is birationally equivalent to the plane curve :

$$w^2 = (t^2 - \frac{1 - k'}{1 + k'})(t^2 - \frac{1 + k'}{1 - k'}), \quad (t, w) \in \mathbb{C}^2. \quad (44)$$

With $\mathbb{P}_{(1,1,2)}^2$ as a compactification of \mathbb{C}^2 via the identity:

$$[1, t, iw] = [y_1, y_2, y_3] \in \mathbb{P}_{(1,1,2)}^2, \quad (45)$$

the equation (44) can be rewritten as

$$W_{k'} \simeq Y_\epsilon : y_1^4 + y_2^4 + y_3^2 - \epsilon y_1^2 y_2^2 = 0, \quad [y_1, y_2, y_3] \in \mathbb{P}_{(1,1,2)}^2, \quad (\epsilon = 2 \frac{1+k'^2}{1-k'^2}). \quad (46)$$

Now the branched points of Ψ are given by zeros of y_1 or y_2 :

$$\begin{aligned} p : \quad (t, \lambda) = (\infty, 0) &\longleftrightarrow [y_1, y_2, y_3] = [0, 1, i]; \\ p' : \quad (t, \lambda) = (\infty, \infty) &\longleftrightarrow [y_1, y_2, y_3] = [0, 1, -i]; \\ q : \quad (t, \lambda) = (0, k') &\longleftrightarrow [y_1, y_2, y_3] = [1, 0, -i]; \\ q' : \quad (t, \lambda) = (0, \frac{1}{k}) &\longleftrightarrow [y_1, y_2, y_3] = [1, 0, i]. \end{aligned}$$

The Boltzmann weights a, b, c, d in Ising model can be regarded as sections on $W_{k'}$ [13]:

$$\begin{aligned} a &\in \Gamma(W_{k'}, \mathcal{O}(q)), \quad b \in \Gamma(W_{k'}, \mathcal{O}(q')), \\ c &\in \Gamma(W_{k'}, \mathcal{O}(p')), \quad d \in \Gamma(W_{k'}, \mathcal{O}(p)). \end{aligned}$$

Over a 4-fold cover $\widetilde{W}_{k'}$ of $W_{k'}$, the above a, b, c, d are linearly equivalent and satisfy the quadratic relations, which give rise to the equations of $\widetilde{W}_{k'}$ in \mathbb{P}^3 :

$$\widetilde{W}_{k'} : \begin{cases} a^2 + k'b^2 = kd^2 \\ k'a^2 + b^2 = kc^2 \end{cases} \quad \text{for } [a, b, c, d] \in \mathbb{P}^3, \quad (47)$$

equivalently,

$$\begin{cases} ka^2 + k'c^2 = d^2 \\ kb^2 + k'd^2 = c^2 \end{cases} \quad \text{for } [a, b, c, d] \in \mathbb{P}^3. \quad (48)$$

It is known that the variables a, b, c, d have the Jacobi elliptic function parametriaization [3] [15]. Here we follow the formulation in [4] [13] by expressing those variables via the prime form $\vartheta_1(z) (= \vartheta_1(z, \tau))$ of the elliptic curve. By formulae in [13] pp. 632- 633², together with (11), a, b, c, d have the following expression:

$$a^2 : b^2 : c^2 : d^2 = -e^{2\pi iz} \vartheta_1(z + \frac{\tau - 1}{4})^2 : -\vartheta_1(z + \frac{-\tau + 1}{4})^2 : \vartheta_1(z + \frac{-\tau - 1}{4})^2 : e^{2\pi iz} \vartheta_1(z + \frac{\tau + 1}{4})^2$$

with

$$\begin{aligned} k' &= -i \frac{\vartheta_2(0, \tau)^2}{\vartheta_4(0, \tau)^2} \quad (= \frac{-e^{-\pi i \tau/2} \vartheta_1(\frac{1}{2}, \tau)^2}{i \vartheta_1(\frac{\tau}{2}, \tau)^2}), \\ k &= \frac{\vartheta_3(0, \tau)^2}{\vartheta_4(0, \tau)^2}. \end{aligned} \quad (49)$$

Within constant factors, the variables a, b, c, d are proportional to the four Jacobi functions $\vartheta_2, \vartheta_4, \vartheta_3, \vartheta_1$ with the same argument. In fact by using (4), one has the following expression:

$$a^2 : b^2 : c^2 : d^2 = i\vartheta_2(z, \tau)^2 : \vartheta_4(z, \tau)^2 : \vartheta_3(z, \tau)^2 : -i\vartheta_1(z, \tau)^2.$$

²A constant was missing in the expression of k' in [13] pp. 632. The correct formula for k' is as follows:

$$k' = \frac{-e^{-\pi i(\rho_1 + \dots + \rho_g)} \vartheta[\frac{\bar{\delta}}{\bar{\nu}}](\varepsilon, \tau)^N}{i^g \vartheta[\frac{\bar{\delta}}{\bar{\nu}}](\varrho, \tau)^N}.$$

Then the equations (47) (48) are equivalent to the 2nd-4th and 3rd-1st relations in (10) for Jacobi elliptic functions. By Sect. 3 of [13], the variables λ, t are related to a, b, c, d by

$$\lambda = \frac{d^2}{c^2}, \quad t = \frac{ab}{cd},$$

hence we obtain theta function representations of λ and t :

$$\begin{aligned} \lambda &= \frac{e^{2\pi iz}\vartheta_1(z+\frac{\tau+1}{4},\tau)^2}{\vartheta_1(z-\frac{\tau+1}{4},\tau)^2} = -i\frac{\vartheta_1(z,\tau)^2}{\vartheta_3(z,\tau)^2}, \\ t &= \frac{\vartheta_1(z+\frac{\tau-1}{4},\tau)\vartheta_1(z+\frac{-\tau+1}{4},\tau)}{\vartheta_1(z-\frac{\tau+1}{4},\tau)\vartheta_1(z+\frac{\tau+1}{4},\tau)} = -i\frac{\vartheta_2(z,\tau)\vartheta_4(z,\tau)}{\vartheta_3(z,\tau)\vartheta_1(z,\tau)}. \end{aligned} \quad (50)$$

Note that the Picard-Fuchs equation for the Ising-family (46) is equivalent to that of X_9 -family. In fact, the equation is derived by Dwork-Griffith-Katz reduction method from residuum expression of the period:

$$\hat{\omega}(s_0, s_1, s_2) = \int_{\gamma} \int_{\Gamma_i} \frac{y_1 dy_2 \wedge dy_3 - y_2 dy_1 \wedge dy_3 + \frac{1}{2} y_3 dy_1 \wedge dy_2}{s_1 y_1^4 + s_2 y_2^4 + y_3^2 - s_0 y_1^2 y_2^2},$$

where γ is a small circle in $\mathbb{P}^2_{(1,1,2)}$ normal to the elliptic curve, Γ_i are 1-circles on the curve. The above integral is also expressed by:

$$\hat{\omega}(1, 1, \epsilon) = \frac{-1}{2} \int_{\Gamma_i} \frac{dt}{w}, \quad \epsilon = 2 \frac{1 + k'^2}{1 - k'^2}$$

where (t, w) are the coordinates of $W_{k'}$ in (44). It is known that $\hat{\omega}(s_0, s_1, s_2)$ satisfies the following equations:

$$\begin{cases} (s_0 \frac{\partial}{\partial s_0} + s_1 \frac{\partial}{\partial s_1} + s_2 \frac{\partial}{\partial s_2} + \frac{1}{2})\hat{\omega} = 0, \\ (s_1 \frac{\partial}{\partial s_1} - s_2 \frac{\partial}{\partial s_2})\hat{\omega} = 0, \\ (\frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2} - \frac{\partial^2}{\partial s_0^2})\hat{\omega} = 0. \end{cases} \quad (51)$$

By the ansatz

$$\hat{\omega}(s_0, s_1, s_2) = \frac{1}{\sqrt{s_0}} \omega(\zeta), \quad \zeta := \frac{s_1 s_2}{s_0^2} = \frac{1}{\epsilon^2},$$

the equation (51) is brought into the form

$$[4(4\zeta - 1)(\zeta \frac{\partial}{\partial \zeta})^2 + 16\zeta^2 \frac{\partial}{\partial \zeta} + 3\zeta] \omega(\zeta) = 0, \quad (52)$$

which has 3 regular singular points at

$$\zeta = 0, \infty, \frac{1}{4}.$$

By the change of coordinates,

$$z = \frac{-\zeta}{16} + \frac{1}{64} = \frac{-1}{16\epsilon^2} + \frac{1}{64},$$

the equation (52) becomes

$$[(z \frac{\partial}{\partial z})^2 - 4(4z \frac{\partial}{\partial z} + 3)(4z \frac{\partial}{\partial z} + 1)] \omega(z) = 0,$$

which is the differential operator in Table (I) for X_9 .

6 Jacobi Elliptic Function Parametrization of X_9 -family

In this section we investigate the X_9 -family :

$$(X_9) \quad X_s : f_s(x) = x_1^4 + x_2^4 + x_3^2 - sx_1x_2x_3 = 0, \quad [x_1, x_2, x_3] \in \mathbb{P}_{(1,1,2)}^2.$$

The curve X_s degenerates at

$$s = \infty, \quad 2\sqrt{2}\omega \quad (\omega^4 = 1),$$

and for $s = 2\sqrt{2}\omega$, it becomes the union of two rational curves:

$$x_1^4 + x_2^4 + x_3^2 - 2\sqrt{2}\omega x_1x_2x_3 = (x_3 - \sqrt{2}\omega x_1x_2)^2 + (x_1^2 - \omega^2 x_2^2)^2 = 0.$$

The linear transformation group preserving a generic polynomial $f_s(x)$ is generated by C, R, Σ which are defined by

$$\begin{aligned} C(x_1, x_2, x_3) &= (ix_1, ix_2, -x_3), & R(x_1, x_2, x_3) &= (-x_1, x_2, -x_3), \\ \Sigma(x_1, x_2, x_3) &= (x_2, x_1, x_3). \end{aligned} \quad (53)$$

The subgroup generated by C and R ,

$$\langle C, R \rangle = \{\text{dia.}(\alpha_1, \alpha_2, \alpha_3) \mid \alpha_1^4 = \alpha_2^4 = \alpha_3^2 = \alpha_1\alpha_2\alpha_3 = 1\} \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_2,$$

acts on $\mathbb{P}_{(1,1,2)}^2$ as an order 2 group induced by R . Denote $r_s, \tilde{\sigma}_s$ the restriction of R and Σ on X_s ,

$$r_s : X_s \longrightarrow X_s, \quad \tilde{\sigma}_s : X_s \longrightarrow X_s.$$

Then r_s is an order 2 translation and $\tilde{\sigma}_s$ is an involution of X_s with

$$r_s \cdot \tilde{\sigma}_s = \tilde{\sigma}_s \cdot r_s.$$

The zeros of x_i 's in X_s are given by

$$\text{zero}(x_1) = \{[0, 1, \pm 1]\}; \quad \text{zero}(x_2) = \{[1, 0, \pm 1]\}; \quad \text{zero}(x_3) = \{[1, \eta, 0] \mid \eta^4 = -1\},$$

each of which is stable under the action of r_s . Via the the projection

$$p : \mathbb{P}_{(1,1,2)}^2 \longrightarrow \mathbb{P}_{(1,1,2)}^2 / \langle R \rangle,$$

the coordinates x_i 's of $\mathbb{P}_{(1,1,2)}^2$ give rise to sections ξ_i 's on $\mathbb{P}_{(1,1,2)}^2 / \langle R \rangle$ with $p^*(\xi_i) = x_i$. We have a family of elliptic curves in $\mathbb{P}_{(1,1,2)}^2 / \langle R \rangle$:

$$\Xi_s : \quad \xi_1^4 + \xi_2^4 + \xi_3^2 - s\xi_1\xi_2\xi_3 = 0, \quad [\xi_1, \xi_2, \xi_3] \in \mathbb{P}_{(1,1,2)}^2 / \langle R \rangle.$$

The restriction of p defines a 2-fold cover of elliptic curves:

$$p_s : X_s \longrightarrow \Xi_s := X_s / \langle r_s \rangle.$$

Let σ_s be the involution of Ξ_s induced by $\tilde{\sigma}_s$,

$$\begin{array}{ccc} X_s & \xrightarrow{\tilde{\sigma}_s} & X_s \\ \downarrow & & \downarrow \\ \Xi_s & \xrightarrow{\sigma_s} & \Xi_s, \end{array}$$

and denote the zeros of ξ_i 's in Ξ_s by

$$\text{zero}(\xi_1) = \{p\} ; \quad \text{zero}(\xi_2) = \{p'\} ; \quad \text{zero}(\xi_3) = \{p_3, p_4\}.$$

We have

$$2p \sim 2p' , \quad p + p' \sim p_3 + p_4 ; \quad \text{and} \quad \xi_1^2, \xi_2^2 \in \Gamma(\Xi_s, \mathcal{O}(2p)) , \quad \xi_1 \xi_2, \xi_3 \in \Gamma(\Xi_s, \mathcal{O}(p_3 + p_4)) .$$

The elements p, p', p_3, p_4 determine the equation of Ξ_s by the following lemma. However as we shall see it later on, much efforts are required in order to obtain some theta function forms of ξ_i 's.

Lemma 5. Let \mathbb{E} be an elliptic curve with two elements p, p' in it fixed by an involution θ . Let m be the order 2 translation of \mathbb{E} with $m(p) = p'$. Then

(i) There exist sections

$$f_1 \in \Gamma(\mathbb{E}, \mathcal{O}(p)) , \quad f_2 \in \Gamma(\mathbb{E}, \mathcal{O}(p'))$$

such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{E} & \simeq & \mathbb{E} & \xrightarrow{m} & \mathbb{E} \\ \downarrow & & \downarrow \Psi & & \downarrow \Psi \\ \mathbb{E}/\langle \theta \rangle & \simeq & \mathbb{P}^1 & \xrightarrow{m_0} & \mathbb{P}^1 , \end{array}$$

where $\Psi(x) = [f_1^2(x), f_2^2(x)]$ and $m_0([\xi, \eta]) = [\eta, \xi]$.

(ii) Let p_3 be an element of \mathbb{E} with $\Psi(p_3) = [1, i]$, and $p_4 := m(\theta(p_3))$. Then $p_3 \neq p_4$,

$$4p \sim 4p' \sim 2p_3 + 2p_4 \sim p + p' + p_3 + p_4 ,$$

and for some $s \in \mathbb{C}$ and $f_3 \in \Gamma(\mathbb{E}, \mathcal{O}(p_3 + p_4))$ with $\text{zero}(f_3) = p_3 + p_4$, the following relation holds :

$$f_1^4 + f_2^4 + f_3^2 = sf_1 f_2 f_3 \in \Gamma(\mathbb{E}, \mathcal{O}(4p)) .$$

Proof. By identifying $\mathbb{E}/\langle \theta \rangle$ with \mathbb{P}^1 , we may assume the map

$$\Psi : \mathbb{E} \longrightarrow \mathbb{P}^1 = \mathbb{E}/\langle \theta \rangle$$

sends p, p' to $[0, 1], [1, 0]$ respectively. Since m commutes with θ , there is an order 2 automorphism m_0 of \mathbb{P}^1 which interchanges the elements $[0, 1]$ and $[1, 0]$, hence for some coordinate system of \mathbb{P}^1 , m_0 is described by

$$m_0([\xi, \eta]) = [\eta, \xi].$$

Then the sections f_1, f_2 in (i) are easily obtained. For $x, y \in \mathbb{E}$, one has

$$y = m(x) \iff x + p' \sim p + y .$$

By the definition of p_3 and p_4 , we have the linearly equivalent relations:

$$2p + p' \sim p_3 + \theta(p_3) + p' \sim p + p_3 + p_4 ,$$

hence

$$p + p' \sim p_3 + p_4 .$$

Then the equivalent relations in (ii) follow immediately. By $\Psi(\theta(p_3)) = \Psi(p_3)$, we have

$$\Psi(p_4) = \Psi(m(\theta(p_3))) = m(\Psi(\theta(p_3))) = m([1, i]) = [1, -i] .$$

Therefore $p_3 \neq p_4$, and

$$f_1^4(p_j) + f_2^4(p_j) = 0 \text{ for } j = 3, 4.$$

Let f_3 be a section in $\Gamma(\mathbb{E}, \mathcal{O}(p_3 + p_4))$ with

$$\text{zero}(f_3) = p_3 + p_4, \quad f_2^4(p) + f_3^2(p) = 0.$$

Both sections $f_1^4 + f_2^4 + f_3^2$ and $f_1 f_2 f_3$ in $\Gamma(\mathbb{E}, \mathcal{O}(4p))$ vanish at p_3, p_4 and p , hence they are proportional by a non-zero constant. Therefore we obtain (ii). \square

Consider the birational map

$$\phi : \mathbb{P}_{(1,1,2)}^2 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad [x_1, x_2, x_3] \mapsto ([x_1, x_2], [x_3, x_1 x_2]),$$

whose fundamental locus consists of 3 elements, defined by two of coordinates x_i 's being zero. Outside the fundamental locus, ϕ defines an isomorphism. The morphism R of $\mathbb{P}_{(1,1,2)}^2$ in (53) induces the morphism \tilde{R} ,

$$\tilde{R} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad ([y_1, y_2], [y_3, y_4]) \mapsto ([y_1, -y_2], [y_3, y_4]),$$

with the commutative diagram:

$$\begin{array}{ccc} \mathbb{P}_{(1,1,2)}^2 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1 \\ R \downarrow & & \downarrow \tilde{R} \\ \mathbb{P}_{(1,1,2)}^2 & \xrightarrow{\phi} & \mathbb{P}^1 \times \mathbb{P}^1. \end{array}$$

Hence ϕ induces a birational morphism between $\mathbb{P}_{(1,1,2)}^2 / \langle R \rangle$ and $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathbb{P}_{(1,1,2)}^2 / \langle R \rangle \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1, \quad [\xi_1, \xi_2, \xi_3] \mapsto ([\xi_1^2, \xi_2^2], [\xi_3, \xi_1 \xi_3]),$$

under which Ξ_s is embedded into $\mathbb{P}^1 \times \mathbb{P}^1$. The coordinates of Ξ_s in $\mathbb{P}^1 \times \mathbb{P}^1$ are now given by

$$\begin{aligned} \Psi : \Xi_s &\longrightarrow \mathbb{P}^1, \quad x \mapsto [\xi_1^2(x), \xi_2^2(x)], \quad \lambda(x) := \frac{\xi_1^2}{\xi_2^2}(x); \\ \Pi' : \Xi_s &\longrightarrow \mathbb{P}^1, \quad x \mapsto [\xi_3(x), \xi_1 \xi_2(x)], \quad u(x) := \frac{\xi_3}{\xi_1 \xi_2}(x), \end{aligned} \tag{54}$$

which defines a plane curve birational to Ξ_s :

$$u^2 - su = -(\lambda + \frac{1}{\lambda}), \quad (u, \lambda) \in \mathbb{C}^2.$$

By the relation

$$\Pi' \sigma_s = \Pi'$$

and the commutative diagram:

$$\begin{array}{ccc} \Xi_s & \xrightarrow{\sigma_s} & \Xi_s \\ \Psi \downarrow & & \downarrow \Psi \\ \mathbb{P}^1 & \xrightarrow{\lambda \mapsto \lambda^{-1}} & \mathbb{P}^1, \end{array}$$

the morphism Π' is equivalent to the projection,

$$\Xi_s \longrightarrow \Xi_s / \langle \sigma_s \rangle,$$

and for some involution θ_s of Ξ_s , the following relations hold:

$$\Psi\theta_s = \Psi, \quad \theta_s\sigma_s = \sigma_s\theta_s.$$

It is easy to see that the branched data of Ψ and Π' are given by

$$\begin{aligned} \text{Branched points of } \Psi: \quad (u, \lambda) &= (\frac{s}{2}, k'), \quad (\frac{s}{2}, \frac{1}{k'}), \quad (\infty, 0), \quad (\infty, \infty); \\ \text{Branched points of } \Pi': \quad (u, \lambda) &= (\frac{s \pm \sqrt{s^2-8}}{2}, 1), \quad (\frac{s \pm \sqrt{s^2+8}}{2}, -1), \end{aligned}$$

where k' is defined by

$$k' + \frac{1}{k'} = \frac{s^2}{4}.$$

Change the variable of \mathbb{P}^1 from u to t via

$$t = \frac{2}{\sqrt[4]{s^4-64}}(u - \frac{s}{2}) \quad (= \sqrt{\frac{k'}{k^2}}[u - (k' + \frac{1}{k'})^{1/2}]), \quad (55)$$

and define the morphism

$$\Pi : \Xi_s \longrightarrow \mathbb{P}^1, \quad \Pi(x) := t(\Pi'(x)). \quad (56)$$

The branched locus Π becomes

$$t = \pm \sqrt{\frac{1-k'}{1+k'}}, \quad \pm \sqrt{\frac{1+k'}{1-k'}}.$$

With Ψ in (54) and Π in (56), one may identify Ξ_s with the curve $W_{k'}$ (42), which is the same as Y_ϵ (46). By (54) (55) and (45), the following elliptic curves are isomorphic:

$$\begin{array}{ccccccc} \Xi_s & \simeq & W_{k'} & \simeq & Y_\epsilon \\ [\xi_1, \xi_2, \xi_3] & \longleftrightarrow & (t, \lambda) & \longleftrightarrow & [y_1, y_2, y_3] \end{array} \quad (57)$$

with the correspondences:

$$\begin{aligned} (t, \lambda) &= \left(\frac{2\xi_3 - s\xi_1\xi_2}{\sqrt[4]{s^4-64}\xi_1\xi_2}, \frac{\xi_1^2}{\xi_2^2} \right) \\ [y_1, y_2, y_3] &= \left[\frac{\sqrt{s^4-64}}{2}\xi_1\xi_2, \xi_3 - \frac{s}{2}\xi_1\xi_2, i(\xi_1^4 - \xi_2^4) \right], \end{aligned}$$

where parameters s, k' and ϵ are related by

$$\frac{s^2}{4} = k' + \frac{1}{k'}, \quad \epsilon = 2\frac{1+k'^2}{1-k'^2} = \frac{2s^2}{\sqrt[4]{s^4-64}}.$$

Note that for the X_9 -family $\{X_s\}_s$ and the Ising family $\{W_{k'}\}_{k'}$ (46), one has

$$\begin{aligned} X_s \simeq X_{s_1} &\iff s_1 = \omega s \quad (\omega^4 = 1); \\ W_{k'} \simeq W_{k'_1} &\iff k'_1 = \pm k', \pm k'^{-1}. \end{aligned}$$

Hence the variable z ,

$$z := s^{-4} = \frac{-1}{16\epsilon^2} + \frac{1}{64} = \frac{k'^2}{16(1+k'^2)^2}, \quad (58)$$

is considered as the moduli parameter of the isomorphic classes of elliptic curves either in X_9 -family, or in Ising family. According to the discussion in Sect. 5, one may identify $W_{k'}$ with $\mathbb{E}_{\tau,1}$ where k', τ satisfy the relation (49). Using (11), we have

$$z(\tau) = \frac{-\vartheta_2(0, \tau)^4 \vartheta_4(0, \tau)^4}{16(\vartheta_3(0, \tau)^8 - 4\vartheta_2(0, \tau)^4 \vartheta_4(0, \tau)^4)}, \quad (59)$$

hence

$$s(\tau) = 2e^{\frac{3\pi i}{4}} \frac{\sqrt{\vartheta_2(0, \tau)^4 - \vartheta_4(0, \tau)^4}}{\vartheta_2(0, \tau) \vartheta_4(0, \tau)}, \quad \frac{\sqrt[4]{s(\tau)^4 - 64}}{2} = e^{\frac{\pi i}{4}} \frac{\vartheta_3(0, \tau)^2}{\vartheta_2(0, \tau) \vartheta_4(0, \tau)}.$$

Theorem 4. With x_i, ξ_i ($1 \leq i \leq 3$) the coordinates of elliptic curves X_s, Ξ_s in the X_9 -family, p_s the morphism between them, and $r_s, \tilde{\sigma}_s$ the automorphisms of X_s as before. For $\tau \in \mathbb{H}$, define $s(\tau)$ by

$$s(\tau) = 2e^{\frac{3\pi i}{4}} \frac{\sqrt{\vartheta_2(0, \tau)^4 - \vartheta_4(0, \tau)^4}}{\vartheta_2(0, \tau) \vartheta_4(0, \tau)}.$$

Then the above data for X_s, Ξ_s have the following realization in complex tori:

$$\begin{aligned} X_s &= \mathbb{E}_{\tau+1,2}, & \Xi_s &= \mathbb{E}_{\tau,1}, & p_s : X_s &\longrightarrow \Xi_s, [z] \mapsto [z]; \\ r_s : \mathbb{E}_{\tau+1,2} &\longrightarrow \mathbb{E}_{\tau+1,2}, & [z] &\mapsto [z+1]; \\ \tilde{\sigma}_s : \mathbb{E}_{\tau+1,2} &\longrightarrow \mathbb{E}_{\tau+1,2}, & [z] &\mapsto [-z + \frac{\tau+1}{2}], \end{aligned}$$

with ξ_i 's given by

$$\begin{aligned} \xi_1 &= \vartheta_1(z, \tau), \\ \xi_2 &= e^{\frac{\pi i}{4}} \vartheta_3(z, \tau), \\ \xi_3 &= \frac{\vartheta_3(0, \tau)^2}{\vartheta_2(0, \tau) \vartheta_4(0, \tau)} \vartheta_2(z, \tau) \vartheta_4(z, \tau) - \frac{\sqrt{\vartheta_2(0, \tau)^4 - \vartheta_4(0, \tau)^4}}{\vartheta_2(0, \tau) \vartheta_4(0, \tau)} \vartheta_1(z, \tau) \vartheta_3(z, \tau), \end{aligned}$$

and through the Heisenberg group action on entire functions, one has the projective representation of $\langle r_s, \tilde{\sigma}_s \rangle$ on x_i 's :

$$\begin{aligned} r_s : (x_1, x_2, x_3) &\mapsto (-x_1, x_2, x_3), \\ \tilde{\sigma}_s : (x_1, x_2, x_3) &\mapsto (-e^{\pi i/4} x_2, -e^{\pi i/4} x_1, e^{\pi i/2} x_3). \end{aligned}$$

Proof. According to the discussion we have before, with $s = s(\tau)$ one has the identification:

$$\Xi_s = \mathbb{E}_{\tau,1},$$

and the rational functions Ψ, Π in (54) (56) equal to

$$\begin{aligned} \Psi : \mathbb{E}_{\tau,1} &\longrightarrow \mathbb{P}^1 = \mathbb{E}_{\tau,1} / \langle \theta \rangle, \quad \text{where} \quad \theta([z]) = [-z]; \\ \Pi : \mathbb{E}_{\tau,1} &\longrightarrow \mathbb{P}^1 = \mathbb{E}_{\tau,1} / \langle \sigma \rangle, \quad \text{where} \quad \sigma([z]) = [-z + \frac{\tau+1}{2}]. \end{aligned}$$

Note that the above σ is identified with the automorphism σ_s of Ξ_s , which can be lifted to the automorphism $\tilde{\sigma}_s$ on X_s . Write $X_s = \mathbb{C}/L$ for some index 2 sublattice L of $\mathbb{Z}\tau + \mathbb{Z}$. The morphism p_s is given by the natural projection. On the universal covering space \mathbb{C} of X_s , the affine map

$$z \mapsto -z + \frac{\tau+1}{2}, \quad z \in \mathbb{C},$$

induces the order 2 automorphism $\tilde{\sigma}_s$ on X_s , hence the element $\tau+1$ is in the lattice L . This implies $X_s = \mathbb{E}_{\tau+1,2}$ with $r_s, \tilde{\sigma}_s$ described in the theorem. The Jacobi elliptic function parametrization of

ξ_i 's now follows from (50) (57), and the action of r_s, \tilde{r}_s on x_i 's is obtained by formulae in Sect. 2. \square

Now the formula for z ($= s^{-4}$) with $s = s(\tau)$ can be derived from the above theorem as follows:

Theorem 5 . The function $z(\mathbf{q})$ for X_9 -family of Table (I) is given by

$$\begin{aligned} z(\mathbf{q}) &= \frac{-\vartheta_2(0,2t-1)^4\vartheta_4(0,2t-1)^4}{16\vartheta_3(0,2t-1)^8-64\vartheta_2(0,2t-1)^4\vartheta_4(0,2t-1)^4} \\ &= \frac{\mathbf{q} \prod_{n=1}^{\infty} (1+\mathbf{q}^n)^8}{\prod_{n=1}^{\infty} (1-\mathbf{q}^{2n-1})^{16} + 64\mathbf{q} \prod_{n=1}^{\infty} (1+\mathbf{q}^n)^8}, \quad \mathbf{q} = e^{2\pi it}. \end{aligned}$$

As a consequence, $z(\mathbf{q})$ has an integral \mathbf{q} -expansion with

$$\lim_{\mathbf{q} \rightarrow 0} \frac{z(\mathbf{q})}{\mathbf{q}} = 1.$$

Proof. For $s = s(\tau)$ in Theorem 4, we have

$$X_s = \mathbb{E}_{\tau+1,2} \simeq \mathbb{E}_{t,1}, \quad t = \frac{\tau+1}{2}.$$

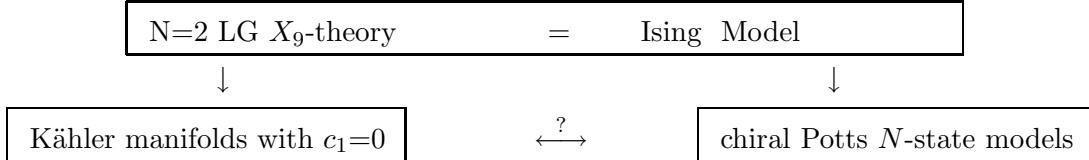
With the theta-constant expression (49) for k' , the relation (58) gives rise the expression of $z(\mathbf{q})$. The same argument as in Theorem 3 shows that t is the variable described in Sect. 3. Therefore we have completed the proof of this theorem. \square

7 Discussion

In this paper, we have focused on the mathematical structure of "counting" functions $z(\mathbf{q})$, and have developed an analysis of constrained Table (I) by means of elliptic theta function representations. Now we are going to discuss another aspect, which is possibly of certain physical relevance. Here the explicitly work out example of X_9 -family has illustrated its close connection with Ising model in statistical mechanics, where we employ the Jacobi elliptic function parametrization of Ising model to the investigation of X_9 -potential. The relation (58) states the parameter z in X_9 -family corresponds to the temperature-like parameter k' of Ising model. One interesting point for the derivation of the function $z(\mathbf{q})$ is that on one hand it is related to Picard-Fuchs equation of the elliptic family, while on the other side with the parametrization for Boltzmann weights of the statistical model, the same result is correctly reproduced. In this setting, the mirror symmetry of X_9 -family is connected to the \mathbb{Z}_2 -cover of elliptic curves in Ising model, which often appears in the theory. Due to the relative simplicity of the models, the quantities involved in our mathematical work usually have interpretations of physical or geometrical meaning, which allows one to compare of their essential structures in an explicit way:

<u>$N = 2$ SUSY LG theory</u>	<u>$2 - \dim.$ exactly solvable model</u>
LG fields	\longleftrightarrow Boltzmann weights
LG superpotential	\longleftrightarrow Yang – Baxter equation
moduli parameter	\longleftrightarrow temperature – like parameter
maximally unipotent area	\longleftrightarrow low temperature region

Though we do not know now whether other models are related in a similar manner, it should be interesting to note that in the work carried out in this article, the mathematical structures of corresponding concepts do share a common feature. The theta function parametrization of Ising model we used here has a direct generalization to chiral Potts N -state models. The naive quantitative indications presented by two physical theories, X_9 and Ising models, encourage us to seek a possible link between Calabi-Yau manifolds and chiral Potts models.



The connection proposed in above diagram is vague. Nevertheless some of symmetries presented in the study of Calabi-Yau spaces resemble those in chiral Potts N -state models. So, we hope some appropriate geometric picture does exist. How to detect this novel phenomena should be of merit for further investigation.

Acknowledgements

The author would like to thank A. Klemm, B. H. Lian, and S. T. Yau for discussions on mirror symmetry, and also to S. S. Lin for discussions on 2-dimensional solvable statistical models.

References

- [1] G Albertini. B. M. McCoy and J.H.H.Perk, Eigenvalue spectrum of the superintegrable chiral Potts model. *Adv. Stud. Pure Math.* 19 (1989) 1-55.
- [2] H. Au-Yang and J.H.H.Perk, Onsager's star-triangle equation: Master key to integrability, *Adv. Stud. Pure Math.* 19 (1989) 57-94.
- [3] R. J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).
- [4] R. J. Baxter, Hyperelliptic function parametrization for chiral Potts model, *Proc. of Intnl Cong. of Math. Kyoto*, 1990.
- [5] R. J. Baxter, V. V. Bazhanov and J.H.H.Perk, Functional relations for transfer matrices of the chiral Potts models, *Intnl. J. Mod. Phys. B* 4 (1990) 803-870
- [6] P. Candelas, X. de la Ossa, P. S. Green and L. Park, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, *Nucl. Phys. B* 359 (1991) 21-74.
- [7] R. C. Gunning, Lectures on modular forms, *Ann. Math. Study*, Princeton Univ. Press, 1962.
- [8] A. Klemm, B. H. Lian, S. S. Roan and S. T. Yau, A note on ODEs from mirror symmetry, CERN-TH. 7369/94, To appear in Functional analysis on the eve of the 21st century-in honor of the 80th birthday of I.M. Gelfand, Birkh\"auser Boston.
- [9] W. Lerche , D. L\"ust and N. P. Warner , Duality symmetries in N=2 Landau-Ginzburg models, CERN-TH. 5504-89.
- [10] B. H. Lian and S. T. Yau, Arithmetic properties of mirror map and quantum coupling, hep-th/9411234.

- [11] D. Mumford, Tata lectures on Theta I , Progress in Math. Vol. 28 Basel: Birkhäuser.
- [12] S. S. Roan, On the Calabi-Yau orbifolds in weighted projective spaces, Intnl. J. Math. (1) 2 (1990) 211-232.
- [13] S. S. Roan, A characterization of "rapidity" curve in the chiral Potts model, Commun. Math. Phys. 145 (1992) 605- 634.
- [14] G. Shimura , Introduction to the arithmetic theory of automorphic functions, Publications of the Math. Society of Japan 11 , Iwanami Shoten Publishers and Princeton Univ. Press.
- [15] L. A. Takhtadzhan and L. D. Faddeev , The quantum method of the inverse problem and the Heisenberg XYZ model, Russian Math. Surveys 34: 5 (1979), 11-68.
- [16] E. T. Whittaker and G. N. Watson, A course of modern analysis, part 2, fourth ed. Cambridge Univ. Press 1962.
- [17] S. T. Yau (Ed.), Essay on mirror manifolds, Intnl. Press, Hong Kong (1992).